Non-commutative Projective Geometry

Daniel Chan reporting on joint work with Adam Nyman

University of New South Wales

April 2009
What is non-commutative projective geometry?

always work over $k = \mathbb{C}$
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Objects of Study

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**E.g.** 1 3-dim Sklyanin algebras. Generic ones are

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A = Skl(a, b, c) := k\langle x_0, x_1, x_2 \rangle / (ax_i x_{i+1} + bx_{i+1} x_i + cx_i^2)_{i \geq 0} \bmod 3
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for generic \((a : b : c) \in \mathbb{P}^2\).
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3. $\dim_k A_n = \left( \frac{n+2}{2} \right) = \dim_k k[x_0, x_1, x_2]_n$. 

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4. $A$ has global dimension 3.

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$$A = \frac{k\langle e, f, h, z \rangle}{(ef - fe - zh, eh - he - 2ze, fh - hf + 2zf, ze - ez, zh - hz, zf - fz)}.$$


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We wish to study these analogues of the polynomial ring in 4 variables as well as their quotients of form $A/(t)$ where $t \in Z(A)_2 \simeq k^2$. 

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Can we study these algebras geometrically?
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Can we study these algebras geometrically?

Need to review some basic commutative algebraic geometry to make sense of this question properly.
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There's a surjective map $\mathbb{P}_X(V) \xrightarrow{f} X$ with fibre $f^{-1}(x) = \mathbb{P}(V_x)$.  

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Let $A = k[x_0, \ldots, x_n]/I$ be homog co-ord ring of proj scheme $Y$. These are closely related.
Links with algebra

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E.g. Points
$y = (1 : y_1 : \ldots : y_n) \in Y \iff f(1, \ldots, y_n) = 0 \quad \forall f \in I$

Defn A point module is a graded $A$-module, generated in degree 0 with $\dim kM_i = 1$, $\forall i \geq 0$.

In fact, get 1-1 correspondence Points of $Y \leftrightarrow$ Point modules of $A$.

Key Point The defn of point modules makes sense for nc graded algebras too. In fact, Artin-Tate-Van den Bergh (1990) use them crucially in their study of 3-dim Sklyanin algebras to prove the aforementioned facts about them.

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So $y \in Y$ gives rise to graded $A$-module $M = A(Y) = k[x_0, \ldots, x_n]/m_y$ with $\dim_k M_i = 1, \forall i \geq 0$. 
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**Serre’s theorem**

If $A$ is the homog co-ord ring of proj scheme $Y$, then Proj $A \cong$ category of quasi-coherent sheaves on $Y$.

Important e.g. of sheaves on $Y$.
If homog co-ord ring $A = k[x_0, \ldots, x_n]/I$ and $J \lhd k[x_0, \ldots, x_n]$ contains $I$,
then its zeros define a subscheme $Z \subseteq Y$.
The structure sheaf of $Z$ is $\mathcal{O}_Z \leftarrow\rightarrow k[x_0, \ldots, x_n]/J$.

**Special Case**

i) $\mathcal{O}_Y \leftarrow\rightarrow A$

ii) $Z = \text{pt on } Y$ recovers point modules.
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**Special Case** i) \( \mathcal{O}_Y \hookrightarrow A \)
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Mori’s philosophy for studying higher dimensional varieties

Let $Y =$ smooth projective scheme (think manifold)
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Let $Y =$ smooth projective scheme (think manifold)
Try to construct map $f : Y \longrightarrow X$ with $X$ and the fibres of $f$ simpler than $Y$. 

Consider the important invariant, the dualising sheaf 
$\omega_Y := \bigwedge^\text{top} T^* Y$, which is a line bundle on $Y$.

Look for $K$-negative curves on $Y$, i.e. curves $C \subset Y$ s.t. $0 > K \cdot C$.

If they exist, then for an extremal $K$-negative curve $C$, there exists a morphism $f : Y \longrightarrow X$ which contracts a curve $C' \subset Y$ iff $C, C'$ are proportional in $H^2(Y, \mathbb{R})$.

For example, Thm Let $Y$ be a comm smooth proj surface and $C \subset Y$ an extremal $K$-negative curve with $C^2 = 0$, then there’s a morphism $f : Y \longrightarrow X$ to a smooth proj curve $X$, contracting $C$, exhibiting $Y$ as a $P^1$-bundle. In particular, $C \cong P^1$ so $H^1(O_C) = 0$. 

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Daniel Chan reporting on joint work with Adam Nyman
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Is there a nc version of this contraction theorem?

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In 1995, Artin-Stafford showed non-commutative smooth projective curves were essentially commutative. Thus non-commutative surfaces are the focus of research now.
In 1997, Artin conjectured that, "up to birational equivalence", a non-commutative projective surface is either a nc $\text{Proj}$ $P^1$-bundle over a curve or finite over its centre.
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Studied by Yekutieli, Zhang, Van den Bergh since 1990

Let $A = \text{noeth conn graded } k\text{-algebra fin gen in degree 1}$. Define $A^\geq 0$-torsion functor $\Gamma^m : \text{Gr} A \to \text{tors}$ by $\Gamma^m = \lim_{n \to \infty} \text{Hom}_A(A/\geq n, -)$. Let $\omega_A := R\Gamma^m(A) \in D_+(A \otimes k)$. It is well-behaved under certain technical conditions (omitted), in which case we will say that $\omega_A$ is an Auslander balanced dualising complex.

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Gelfand-Kirillov dimension: 
$$gk(M) = \text{growth rate of } f_M(n) := \dim k(\bigoplus_{i \leq n} M_i) = \inf \{c | f_M(n) \leq nc, n \gg 0\}.$$

Canonical dimension: 
$$c.\dim(M) := \max \{i | R_i \Gamma_m(M) \neq 0\}.$$

On $Y = \text{Proj } A,$ we can sensibly define $\dim = gk - 1$ or $c.\dim - 1.$
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Let $A = \text{noeth conn graded } k\text{-algebra fin gen in degree 1}$. There are 2 good notions of dimension of a noeth $M \in \text{Gr } A$.

**Gelfand-Kirillov dimension:**

$$gk(M) = \text{growth rate of } f_M(n) := \dim_k(\bigoplus_{i \leq n} M_i)$$

$$= \inf \{ c | f_M(n) \leq n^c, n \gg 0 \}.$$  

**Canonical dimension:**

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On $Y = \text{Proj } A$, we can sensibly define

$$\dim = gk - 1 \text{ or } c.\dim - 1$$
Let $Y = \text{Proj } A = \text{nc smooth proj surface}$
Intersection theory (I. Mori-Smith)

Let $Y = \text{Proj } A = \text{nc smooth proj surface}$

For $M, N \in \text{mod } Y$, have well-defined intersection pairing

$$M \cdot N := -\sum_{i=0}^{2} (-1)^i \dim \text{Ext}^i_Y(M, N)$$
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**Motivation** If $Y$ is comm, & $C, D \subset Y$ are curves

then $\mathcal{O}_C \cdot \mathcal{O}_D = C.D$
Hilbert schemes (Artin-Zhang)

Let $Y = \text{Proj } A = \text{nc smooth proj } d$-fold

Theorem (loosely stated Artin-Zhang 2001)

For $P \in \text{mod } Y$, there exists a Hilbert scheme $\text{Hilb } P$ parametrising quotients of $P$.

$\text{Hilb } P$ is a countable union of projective schemes which is locally of finite type.

To set up moduli problem, we need the notion of a flat family.

For comm-$k$-algebra $R$ we can define $\text{Mod } Y \otimes R = \text{Proj } A \otimes k \otimes R / \text{tors}$

For $M \in \text{Mod } Y \otimes R$ we can define $M \otimes R^{-} : \text{Mod } R^{-} \rightarrow \text{Mod } Y \otimes R$ and hence flatness over $R$.

Remark: Artin-Tate-Van den Bergh show the Hilbert scheme of point modules on $\text{Skl}(a, b, c)$ is a genus one curve.

They deduce that nc $P^2$s contain a genus one curve.

Daniel Chan reporting on joint work with Adam Nyman
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To set up moduli problem, we need the notion of a flat family. For comm $k$-algebra $R$ can define

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Daniel Chan reporting on joint work with Adam Nyman
Let $Y = nc$ smooth proj surface
Let \( Y = \text{nc smooth proj surface} \)

Say \( M \in \text{mod } Y \) is a \textit{K-non-effective rational curve with self-intersection 0} if

1. \( M \) is a 1-critical quotient of \( O_Y \).
2. \( H^0(M) = k, H^1(M) = 0 \).
3. \( H^0(M \otimes \omega_Y) = 0 \).

When it exists, we wish to find a projective curve \( X \) and a morphism \( f: Y \to X \) i.e. \textit{adjoint functors} \( f^*: \text{Mod } X \to \text{Mod } Y \), \( f_*: \text{Mod } Y \to \text{Mod } X \).
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Let $Y = \text{nc smooth proj surface}$,
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Theorem (C-Nyman)

$X$ is a projective curve which is smooth at the point $p$ corresponding to $M$.

If for every simple 0-dim quotient $P \in \text{mod } Y$ of $M$ we have $M.P = 0$, then there is a morphism $f: Y \to X$ and $f^*O_p = M$.

Idea of proof.
Associated to $X$ is a universal flat family $M \in \text{mod } Y$ of quotients of $O_Y$.
Can define a type of Fourier-Mukai transform $f^* : \text{mod } X \to \text{mod } Y$.

One has to check this is right exact.

Daniel Chan reporting on joint work with Adam Nyman
Let $Y = \text{nc smooth proj surface}$, $M = K$-non-eff rat curve with $M^2 = 0$, $X = \text{comp of Hilb}$ $\mathcal{O}_Y$ containing $M$. 

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Non-commutative Mori contraction

Let $Y = \text{nc smooth proj surface}$,  
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Daniel Chan reporting on joint work with Adam Nyman
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$$f^* : \text{mod } X \xrightarrow{\mathcal{M} \otimes X} \text{mod } Y_X \xrightarrow{\pi^*} \text{mod } Y.$$
Non-commutative Mori contraction

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\( M = K\)-non-eff rat curve with \( M^2 = 0 \),  
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**Theorem (C-Nyman)**

- \( X \) is a projective curve which is smooth at the point \( p \) corresponding to \( M \).
- If for every simple 0-dim quotient \( P \in \text{mod } Y \) of \( M \) we have \( M \cdot P = 0 \), then there is a morphism \( f : Y \longrightarrow X \) and \( f^* \mathcal{O}_p = M \).

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