Algebraic stacks in the representation theory of finite-dimensional algebras

Daniel Chan
joint work with Boris Lerner

University of New South Wales
web.maths.unsw.edu.au/~danielch

October 2015
always work over base field $k$ algebraically closed of char 0.

**Motto**

Moduli stacks are a fruitful way to study non-commutative algebra, because they are a machine to construct functors.

**Plan of talk**

- Recall the variety of representations of a quiver with relations.
- Brief user’s guide to stacks in representation theory.

**Question**

Given a finite dimensional algebra $A$, how do you find an algebraic stack which is derived equivalent to it?

We finally,

- introduce a new moduli stack of “Serre stable representations”, which gives a first approximation to answering this question.
We use the following notation

- quiver $Q = (Q_0 = \text{vertices}, Q_1 = \text{edges})$ without oriented cycles
- $kQ$ the path algebra & $I \triangleleft kQ$ an admissible ideal of relations
- $M = \bigoplus_{v \in Q_0} M_v$ is a (right) $A = kQ/I$-module i.e. a representation of $Q$ with relations $I$.

The **dimension vector** of $M$ is
\[
\dim M = (\dim_k M_v)_{v \in Q_0} \in \mathbb{Z}^{Q_0} \cong K_0(A).
\]
Representation variety

Let’s classify representations with dim vector $\vec{d} = (d_v)$. Consider one such $M$.

- Picking bases i.e. isomorphisms $M_v \simeq k^{d_v}$ gives a unique point of
  $$\text{Rep}(Q, \vec{d}) := \prod_{v \to w \in Q_1} \text{Hom}_k(k^{d_v}, k^{d_w}).$$

- Choice of basis is up to group $GL(\vec{d}) := \prod_{v \in Q_0} GL(d_v)$.

- If $I \neq 0$, then $kQ/I$-modules correspond to some closed subscheme
  $$\text{Rep}(Q, I, \vec{d}) \subseteq \text{Rep}(Q, \vec{d}).$$

- $GL(\vec{d})$ acts on $\text{Rep}(Q, I, \vec{d})$ and orbits correspond to isomorphism classes of modules (with dim vector $\vec{d}$),
- stabilisers correspond to automorphism groups of $M$.
- The diagonal copy of $k^\times$ acts trivially so $PGL(\vec{d}) := GL(\vec{d})/k^\times$ also acts.

*Daniel Chan joint work with Boris Lerner*
Motivating example à la King

\[ Q = \text{Kronecker quiver} \quad v \xrightarrow{\delta} w, \quad \vec{d} = \vec{1} = (1 \quad 1). \]

\[ k \xrightarrow{x} k \in \text{Rep}(Q, \vec{1}) \cong k^2 = \mathbb{A}^2 \]

\[ PGL(\vec{1}) = k^{\times 2}/k^{\times} \cong k^{\times} \text{ acts by scaling, so if we omit } (x, y) = (0, 0) \text{ (explain later) have quotient } \text{Rep}(Q, \vec{1}) - (0, 0))/PGL \cong \mathbb{P}^1. \]

We get a family of modules \( M_{(x:y)} = M_{(x:y),v} \xrightarrow{x} M_{(x:y),w} \)
parametrised by \( (x : y) \in \mathbb{P}^1 \) which gives “the” universal representation

\[ \mathcal{U} = \mathcal{O}_{\mathbb{P}^1} \xrightarrow{x} \mathcal{O}_{\mathbb{P}^1}(1) \]

Interesting Fact

\( \mathcal{U} \) is an \( \mathcal{O}_{\mathbb{P}^1} - A \)-bimodule whose dual \( A^T \mathcal{O}_{\mathbb{P}^1} = \mathcal{H}om_{\mathbb{P}^1}(\mathcal{U}, \mathcal{O}) \) induces inverse derived equivalences

\[ \text{RHom}_{\mathbb{P}^1}(\mathcal{T}, -) : D^b(\mathbb{P}^1) \to D^b(A), \quad - \otimes_A^L \mathcal{T} : D^b(A) \to D^b(\mathbb{P}^1) \]

Daniel Chan joint work with Boris Lerner
Stacks: via categorifying Grothendieck’s functor of points

To generalise this eg, need to enlarge category of schemes. A scheme $X$ is not determined by its $k$-points, but is determined by all its $R$-points ($R$ comm ring). More precisely, it’s determined by

**Functor of points**

the *functor of points* of $X$, which is the covariant functor $h_X = \text{Hom}_{\text{Scheme}}(\text{Spec } (-), X) : \text{CommRing} \to \text{Set}$

so $h_X(R) = \{f : \text{Spec } R \to X\}$

**Remark** Compare with maximal atlas defn of a manifold.
We “categorify” this defn, and let Gpd be the category of groupoids $=$ small categories with all morphisms invertible.

**“Definition” (Stack)**

A *stack* is a pseudo-functor $h : \text{CommRing} \to \text{Gpd} + \text{lots of axioms}$. Think of the isomorphism classes of objects in the category $h(k)$ as the “$k$-points” & the category now remembers automorphisms.
Let $G$ be an algebraic group acting on a $k$-variety $X$.

Want a “stacky” group quotient $[X/G]$ st “$k$-points” are the $G$-orbits $G.x$, & the automorphism group of such a point is $\text{Stab}_G x < G$.

Recall A scheme morphism $\tilde{U} \longrightarrow U$ is a $G$-torsor or $G$-bundle if $G$ acts on $\tilde{U}$ and trivially on $U$, is $G$-equivariant and locally on $U$ is the trivial $G$-torsor $\text{pr} : G \times U \longrightarrow U$.

Motivation There should be a $G$-torsor $\pi : X \longrightarrow [X/G]$ so an object of $f \in [X/G](R)$ gives a Cartesian square

$$
\begin{array}{ccc}
\tilde{U} & \xrightarrow{\phi} & X \\
\downarrow q & & \downarrow \pi \\
U := \text{Spec } R & \xrightarrow{f} & [X/G]
\end{array}
$$

$\implies$ objects of $[X/G](R)$ are pairs $(\phi, q)$ st $q : \tilde{U} \longrightarrow \text{Spec } R$ is a $G$-torsor & $\phi : \tilde{U} \longrightarrow X$ is $G$-equivariant.
Define category of coherent sheaves $\text{Coh}[X/G] = \text{category of } G$-equivariant coherent sheaves on $X$ e.g. if $X$ smooth, $\omega_{[X/G]} := \omega_X$.

Consider case $X = \mathbb{A}^1_x$ & $G = \mu_p = \langle \zeta = \sqrt[p]{1} \rangle$ acts by multn, so action free on $x \neq 0$ but $\text{Stab}_G 0 = \mu_p$.

$k$-points are parametrised by $y = x^p$.

- If $y \neq 0$ then $k[x]/(x^p - y)$ is a simple sheaf on $[X/G]$.
- If $y = 0$, then $k[x]/(x^p)$ is non-split extension of $p$ non-isomorphic simples $k[x]/(x)$ with $\mu_p$-action given by the $p$ characters of $\mu_p$.

**General Fact**

If $\tilde{U} \to U$ is a $G$-torsor, then $[\tilde{U}/G] \simeq U$. Here $[(\mathbb{A}^1_x - 0)/\mu_p] \simeq \mathbb{A}^1_y - 0$.

- $\omega_X = k[x]dx$ & $\omega_{[X/G]} \otimes [X/G]$ permutes the simples with $x = 0$ cyclically.
Families through stacky points

Note there is also a “birational” map $[\mathbb{A}^1_x/\mu_p] \to \mathbb{A}^1_y$. The rational inverse $\phi: \mathbb{A}^1_y - 0 \to [\mathbb{A}^1_x/\mu_p]$ given by

\[
\begin{array}{ccc}
\mathbb{A}^1_x - 0 & \longrightarrow & \mathbb{A}^1_x \\
\downarrow_{x \mapsto x^p = y} & & \\
\mathbb{A}^1_y - 0 & &
\end{array}
\]

Important Phenomenon

You can’t extend $\phi$ to all of $\mathbb{A}^1_y$, except by first passing to to an étale cover of $\mathbb{A}^1_y - 0$ as below.

Have tautological quotient map $\mathbb{A}^1_x \to [\mathbb{A}^1_x/\mu_p]$ defined by

\[
\begin{array}{ccc}
\mu_p \times \mathbb{A}^1_x & \xrightarrow{\text{action}} & \mathbb{A}^1_x \\
\downarrow_{pr} & & \\
\mathbb{A}^1_x & &
\end{array}
\]

This process is called “stable reduction”.

Daniel Chan  joint work with Boris Lerner
Weighted projective lines

Can define stacks via gluing just as for schemes.

Let \( y_1, \ldots, y_n \in \mathbb{P}^1 \) and \( p_1, \ldots, p_n \geq 2 \) be integer weights. There is a stack \( \mathbb{W} = \mathbb{P}^1(\sum p_i y_i) \) and map \( \pi : \mathbb{P}^1(\sum p_i y_i) \to \mathbb{P}^1 \) which is

- an isomorphism away from the \( y_i \),
- locally near \( y_i \), it looks like \([\mathbb{A}^1_x/\mu_{p_i}] \to \mathbb{A}^1_y\)

We call \( \mathbb{P}^1(\sum p_i y_i) \) a weighted projective line.

\( \pi^* \) induces an isomorphism

\[
k^2 = \text{Hom}_{\mathbb{P}^1}(\mathcal{O}, \mathcal{O}(1)) \to \text{Hom}_{\mathbb{W}}(\pi^* \mathcal{O}, \pi^* \mathcal{O}(1)).
\]

If \( f_i \in \text{Hom}_{\mathbb{W}}(\pi^* \mathcal{O}, \pi^* \mathcal{O}(1)) \) corresponds to \( y_i \), then

\[
\text{coker}\left(f_i : \pi^* \mathcal{O} \to \pi^* \mathcal{O}(1)\right)
\]

is the non-split extension of \( p_i \) non-isomorphic simples on previous slide.
Factorising $f_i$ into $p_i$ inclusions gives

\[ \mathcal{O}\left(\frac{y_1}{p_1}\right) \longrightarrow \mathcal{O}\left(\frac{2y_1}{p_1}\right) \longrightarrow \ldots \longrightarrow \mathcal{O}\left(\frac{(p_1-1)y_1}{p_1}\right) \]

\[ \mathcal{O}\left(\frac{y_2}{p_2}\right) \longrightarrow \mathcal{O}\left(\frac{2y_2}{p_2}\right) \longrightarrow \ldots \longrightarrow \mathcal{O}\left(\frac{(p_2-1)y_2}{p_2}\right) \]

\[ \mathcal{O}\left(\frac{y_n}{p_n}\right) \longrightarrow \mathcal{O}\left(\frac{2y_n}{p_n}\right) \longrightarrow \ldots \longrightarrow \mathcal{O}\left(\frac{(p_n-1)y_n}{p_n}\right) \]

**Thm (Geigle-Lenzing)** The above is a tilting bundle on $\mathbb{P}^1(\sum p_i y_i)$ with endomorphism ring the corresponding canonical algebra.
Moduli stack of isomorphism classes of $A = kQ/I$-modules

Fix dim vector $\vec{d} \in K_0(A)$. There's a stack $\text{Iso}(A, \vec{d})$ with $k$-points the iso classes of $A$-modules dim vector $\vec{d}$ & automorphisms $=$ module automorphisms.

$\text{Iso}(A, \vec{d})(R) =$ category of $(R, A)$-modules $\mathcal{M} = \oplus \mathcal{M}_v$, with

- $\mathcal{M}_v$ loc free rank $d_v/R$,
- Morphisms $=$ bimodule isomorphism

Important Facts

- $\text{Iso}(A, \vec{d}) \simeq [\text{Rep}(Q, I, \vec{d})/\text{GL}(\vec{d})]$.
- Tautologically, there is a universal $A$-module $\mathcal{U} = \oplus \mathcal{U}_v$ over $\text{Iso}(A, \vec{d})$.

Note These will never be weighted projective lines because all modules have $k^\times$ in their automorphism group!

Daniel Chan joint work with Boris Lerner
Rigidified moduli stack of $A$-modules

We *rigidify* the stack to remove this common copy of $k^\times$. Define (when some $d_v = 1$ else need stackification)

$\text{RigIso}(A, \vec{d})(R)$ has same objects as $\text{Iso}(A, \vec{d})(R)$, but

- a morphism in $\text{Hom}(\mathcal{M}, \mathcal{N})$ is an equivalence class of
  $(R, A)$-bimodule isomorphisms $\psi : \mathcal{M} \longrightarrow L \otimes_R \mathcal{N}$ where $L$ is a line bundle on $R$,
- $\psi : \mathcal{M} \longrightarrow L \otimes_R \mathcal{N}, \psi' : \mathcal{M} \longrightarrow L' \otimes_R \mathcal{N}$ are equivalent if there’s an iso $l : L \longrightarrow L'$ st $\psi' = (l \otimes 1)\psi$.

**Important Facts**

- $\text{RigIso}(A, \vec{d}) \simeq [\text{Rep}(Q, I, \vec{d})/\text{PGL}(\vec{d})]$.
- Tautologically, there is a universal $A$-module $\mathcal{U} = \bigoplus \mathcal{U}_v$ over $\text{RigIso}(A, \vec{d})$, unique up to line bundle.
Assume now gl. dim $A < \infty$ & write $DA = \text{Hom}_k(A, k)$.

Recall we have a Serre functor $\nu = - \otimes_A^L DA$ on $D_{fg}^b(A)$. Define $\nu_d = \nu \circ [-d]$.

Given a $k$-point of $\text{RigIso}(A, \vec{d})$ i.e. $A$-module $M$, $\nu_d M$ may or may not define a $k$-point of $\text{RigIso}(A, \vec{d})$.

**Proposition**

The locus of modules where it does, defines a locally closed substack $\text{RigIso}(A, \vec{d})^0$ of $\text{RigIso}(A, \vec{d})$. It is open if $d = \text{pd } DA$ or $\text{pd } DA - 1$.

We hence obtain a partially defined self-map

$$\nu_d : \text{RigIso}(A, \vec{d})^0 \longrightarrow \text{RigIso}(A, \vec{d})$$
The Serre stable moduli stack

The *Serre stable moduli stack* $\text{RigIso}(A, \vec{d})^S$ is the fixed point stack i.e. fibre product

$$
\begin{array}{ccc}
\text{RigIso}(A, \vec{d})^S & \longrightarrow & \text{RigIso}(A, \vec{d})^0 \\
\downarrow & & \downarrow \Gamma_{\nu d} \\
\text{RigIso}(A, \vec{d}) & \longrightarrow & \text{RigIso}(A, \vec{d}) \times \text{RigIso}(A, \vec{d})
\end{array}
$$

The category of $k$-points $\text{RigIso}(A, \vec{d})^S(k)$ has

- **Objects**: isomorphisms $M \xrightarrow{\sim} \nu_d M$ where $M$ is an $A$-module dim vector $\vec{d}$
- **Morphisms**: diagrams of isomorphisms which commute up to scalar

$$
\begin{array}{ccc}
M & \longrightarrow & \nu_d M \\
\theta \downarrow & & \nu_d \theta \\
N & \longrightarrow & \nu_d N
\end{array}
$$

Objects of $\text{RigIso}(A, \vec{d})^S(R)$ are $(R, A)$-bimodule isomorphisms $M \simeq L \otimes_R M \otimes_A^L DA[-d]$, where $L$ is a line bundle.

Daniel Chan  joint work with Boris Lerner
Serre stability alters points: eg Kronecker algebra

\[ Q = \text{Kronecker quiver} \quad \xrightarrow{v \to w}, \quad \vec{d} = \vec{1} = (1 \quad 1). \quad A = kQ, \quad d = 1. \]

\[ M : \quad k \xrightarrow{0} k \]

has a projective summand \( 0 \xrightarrow{} k \) so \( M \not\cong \nu_1 M \)
\[ \implies \text{no corresponding point of RigIso}(A, \vec{1})^S. \]

However, for the universal representation

\[ U = \mathcal{O}_{\mathbb{P}^1} \xrightarrow{x} \mathcal{O}_{\mathbb{P}^1}(1) \]

we have \( U \otimes^L_A DA[-1] \cong \mathcal{O}_{\mathbb{P}^1} \otimes_{\mathbb{P}^1} U \) & in fact

**Proposition**

\[ \text{RigIso}(A, \vec{1})^S \cong \mathbb{P}^1. \]

A similar result holds for the Beilinson algebra derived equivalent to \( \mathbb{P}^d \).
Serre stability alters automorphism groups

\[ A = \text{canonical algebra of } \mathbb{P}^1(3y). \text{ Let } d = 1, \bar{d} = 1. \]

\[ M := \begin{array}{ccc}
    k & \overset{0}{\longrightarrow} & k \\
    \downarrow 0 & & \downarrow 0 \\
    k & \underset{1}{\longrightarrow} & k
\end{array} \]

is the direct sum of a \( \nu_1 \)-orbit corresponding to the 3 simple sheaves at \( y = 0 \).

- automorphisms of \( M \) in RigIso are \( (k^\times)^3/k^\times \cong (k^\times)^2 \).
- automorphisms of \( M \) in RigIso\(^S\) are \( \mu_3 \)!

**Why**

\[ M \overset{\theta}{\longrightarrow} \nu_1 M \]

\[ \theta \in (k^\times)^3 \]

\[ \nu_d \theta, \nu_d \theta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \theta \]

commutes up to scalar \( \iff \theta \) is an e-vector of the permutation matrix.
The $k$-points of $\text{RigIso}^S$

Note $\nu_d$ induces a (shifted) Coxeter transformation on $K_0(A)$.

If $M \in \text{mod } A$ is Serre stable in sense $M \simeq \nu_d M$, then $\vec{d} := \text{dim} M$ is fixed by $\nu_d$. We say $\vec{d}$ is Coxeter stable.

**Proposition**

Let $M$ be a Serre stable module with $\text{dim} M$ minimal Coxeter stable. If $\text{End}_A M$ is semisimple then

- Any two isomorphisms $\theta : M \rightarrow \nu_d M, \theta' : M \rightarrow \nu_d M$ are isomorphic in $\text{RigIso}^S$.

- The automorphism group in $\text{RigIso}^S(k)$ of any such $\theta$ is $\mu_p$ where $p = \text{no. Wedderburn components of } \text{End}_A M$. 

Daniel Chan joint work with Boris Lerner
Some theorems

Theorem (C.-Lerner)

Let $\mathbb{W}$ be a weighted projective line which is \textit{Fano or anti-Fano} i.e. $\omega_\mathbb{W}^{\mp 1}$ is ample or equiv, is not tubular. Let

\begin{itemize}
  \item $\mathcal{T} = \oplus \mathcal{T}_v$ be a basic tilting bundle on $\mathbb{W}$
  \item $A = \text{End}_{\mathbb{W}} \mathcal{T}$.
\end{itemize}

Then $\text{RigIso}(A, \dim \mathcal{T})^S \cong \mathbb{W}$ & $\mathcal{T}$ is dual to the universal representation.

Remark Higher dimensional versions hold.

Theorem (C.-Lerner)

Let $A = \text{canonical algebra}$. Then $\text{RigIso}(A, \mathbb{1})^S$ is a weighted projective line derived equivalent to $A$ & the universal representation is dual to the tilting bundle given earlier.

- Abdelghadir-Ueda have also exhibited weighted projective lines as moduli spaces, but of enriched quiver representations.
- The proof of the derived equivalence is via Bridgeland-King-Reid theory and is independent of Geigle-Lenzing’s.
Reminder on Bridgeland-King-Reid theory

Let $\mathbb{W}$ be a smooth weighted projective variety. Then the set $\Omega$ of simple sheaves is a spanning class for $\text{Coh } \mathbb{W}$.

Let $\mathcal{T}$ be an $(\mathcal{O}_\mathbb{W}, A)$-bimodule for some fin dim algebra $A$ which is left locally free &

$$F = \text{RHom}_\mathbb{W}(\mathcal{T}, -) : D_c^b(\mathbb{W}) \to D_{fg}^b(A)$$

**Theorem (Bridgeland-King-Reid)**

Suppose for all $S, S' \in \Omega$ we have

- $F : \text{Ext}^i_\mathbb{W}(S, S') \to \text{Ext}^i_A(\mathcal{F}S, \mathcal{F}S')$ is an isomorphism, and

- $\nu(\mathcal{F}S) \simeq F(\omega_\mathbb{W} \otimes_\mathbb{W} S)$.

Then $F$ is a derived equivalence.

**Remark** Serre stability condition makes checking the 2nd condition easy.
A fresh look at the canonical algebra $A$

**Step 1 Choose $\vec{d}$**: For $\text{RigIso}^S \neq \emptyset$ need $\vec{d}$ fixed by Coxeter transformation $= \nu_1$ on $K_0(A)$. Use $\vec{d} = \vec{1}$: it works and generates all such vectors if $A$ is non-tubular.

**Step 2 Compute Serre functor on some modules**: eg for

$M := k \xrightarrow{b} k, \quad \nu_1 M := k \xrightarrow{a} k$

Note iso class determined by product $abc$

**Step 3 Guess a universal family/moduli space**:

$k[x] \xrightarrow{x} k[x]$

is a $\mu_3$-equivariant family on $\mathbb{A}^1_x$. See $\text{RigIso}^S \simeq \mathbb{P}^1(3y)$.
Remark on stable reduction in this case

For \( c \in k - 0 \), we get a Serre stable family

\[
M_c := \begin{array}{c}
k \\
\downarrow c \\
\downarrow 1 \\
k \end{array} \rightarrow \begin{array}{c}
k \\
\downarrow 1 \\
k \end{array} \rightarrow \begin{array}{c}
k \\
\downarrow 1 \\
k \end{array}
\]

which does not immediately extend to \( c = 0 \). Need first adjoin \( \sqrt[3]{c} \) to get

\[
M_{\sqrt[3]{c}} := \begin{array}{c}
k \\
\downarrow \sqrt[3]{c} \\
\downarrow \sqrt[3]{c} \\
k \end{array} \rightarrow \begin{array}{c}
k \\
\downarrow \sqrt[3]{c} \\
k \end{array} \rightarrow \begin{array}{c}
k \\
\downarrow 1 \\
k \end{array}
\]

Daniel Chan  joint work with Boris Lerner
Method “works” because Serre stable moduli stack of “skyscraper sheaves” is the tautological moduli problem that recovers many stacks.

Ideally we can apply Bridgeland-King-Reid theory to obtain independently many derived equivalences. Problem is we don’t have many general results about the Serre stable moduli stack e.g. need a stable reduction theorem.

For tame hereditary algebras, the preprojective algebra arises naturally in attempting to construct Serre stable objects.

Case where you insert weights on intersecting divisors fails. Perhaps can be fixed by using the cotangent bundle.