This problem set covers material from lectures 24-27.

1. Let $X$ be the product of $n$ copies of $S^1$. Compute the cohomology ring of $X$ and show it is a $2^n$-dimensional algebra.

2. In this question, we study the cohomology ring of complex projective $n$-space $\mathbb{C}P^n$. Prove that the cohomology ring $H^*(\mathbb{C}P^n; \mathbb{F}) \simeq \mathbb{F}[u]/(u^{n+1})$ where $u \in H^2(\mathbb{C}P^n; \mathbb{F})$. Note that $\mathbb{C}P^n$ is compact and triangulable (can you prove this?).

3. One can show that the homology of real projective space $\mathbb{R}P^n = (\mathbb{R}^{n+1} - 0)/\mathbb{R}^\times$ is $H_i(\mathbb{R}P^n; \mathbb{F}_2) = \mathbb{F}_2$ for $i = 0, \ldots, n$ and is 0 otherwise. Determine its cohomology ring (with co-efficients in $\mathbb{F}_2$). Hence prove the Borsuk-Ulam theorem: given a continuous map $f : S^n \to \mathbb{R}^n$, there is some $x \in S^n$ such that $f(-x) = f(x)$.

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1 by Daniel Chan