1. Let \( R \) be a UFD and consider a factorisation of \( r = p_1p_2\ldots p_m \in R \) into irreducibles \( p_1, \ldots, p_m \). Prove the claim in lecture 28 that the divisors of \( r \) (i.e. \( d \in R \) such that \( d|r \)) are precisely the products of the \( p_j \)'s and their associates.

2. Let \( R \) be a UFD and \( p, q, r \in R \) be such that \( pq = r^3 \). Prove the claim in lecture 30 that if the greatest common divisor of \( p, q \) is 1, then up to associates, \( p, q \) are themselves cubes.

3. Let \( F \) be any field. Show that the usual factor theorem and remainder theorem hold in \( F[x] \). That is, \( \alpha \in F \) is a zero of \( p(x) \in F[x] \) if and only if \( (x-\alpha) \) is a factor of \( p(x) \) and more generally, \( p(x) = (x-\alpha)q(x) + q(\alpha) \) for some \( q(x) \in F[x] \).

4. Which of the following polynomials are prime in \( \mathbb{R}[x] \): i) \( x^2 + 2 \) ii) \( x^4 + 4 \)?

5. Is \( x^2 + 2xy + 3y + 3 \) prime in \( \mathbb{Q}[x] \)? What about \( \mathbb{Q}(x)[y] \) where \( \mathbb{Q}(x) \) is the field of fractions of \( \mathbb{Q}[x] \)?

6. Is \( \mathbb{Z}[x, y]/\langle x^2 + 1 \rangle \) a UFD?

7. In \( (\mathbb{Z}[i])[x] \) find the greatest common divisor of \( 5x^2 - 10x \) and \( (1+2i)x \).

8. Let \( \omega \) be a primitive cube root of unity. Find a minimal polynomial for \( \omega \) (over \( \mathbb{Q} \)) and the degree of the extension \( \mathbb{Q} (\omega)/\mathbb{Q} \).

9. Let \( E/F, K/E \) be field extensions such that \( K/F \) is algebraic. Are \( E/F \) or \( K/E \) algebraic?

10. Is \( \alpha = \sqrt{3} + \sqrt{5} \) algebraic over \( \mathbb{Q} \). If so determine its minimal polynomial (use Eisenstein criterion if you like). Can you find subfields of \( \mathbb{Q}(\alpha) \) other than \( \mathbb{Q} \)?

11. Show rigorously that \( \mathbb{Q}(\sqrt{3}) \neq \mathbb{Q}(\sqrt{3}, \sqrt{5}) \). Hence determine the degree \( [\mathbb{Q}(\sqrt{3}, \sqrt{5}) : \mathbb{Q}] \). Using this or otherwise, show that \( \alpha = \sqrt{3} + \sqrt{5} \) is algebraic over \( \mathbb{Q} \) and determine its minimal polynomial.

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1\textsuperscript{by Daniel Chan}
12. Let $E/F$ be a finite field extension of prime degree. Show that $E$ is a simple extension of $F$.

13. (Not examinable) Show that $\overline{\mathbb{Q}}$ (set of complex numbers algebraic over $\mathbb{Q}$) is countable and so there are uncountably many numbers in $\mathbb{C}$ which are transcendental over $\mathbb{Q}$.

14. Use the Eisenstein criterion to show that $x^3 + 6x + 18$ is irreducible over $\mathbb{Q}[x]$.

15. What’s the algebraic closure of $\mathbb{R}$? Is $\mathbb{C}(x) := K(\mathbb{C}[x])$ algebraically closed?

16. Let $p$ be prime. Show that $1 + x + x^2 + \ldots + x^{p-1}$ is irreducible over $\mathbb{Q}$ by summing the geometric progression and changing variables to $y = x - 1$ or otherwise.

17. Can you construct a regular 7-gon (inscribed in unit circle, say by scaling appropriately) using ruler and compass? Hint: The previous question may help.

18. Let $\alpha = \cos \frac{\pi}{48}$. Show that $\alpha$ is algebraic over $\mathbb{Q}$ and that in fact, its minimal polynomial has degree a power of 2.

19. Show that $\mathbb{F}_2[x]/\langle x^2 + x + 1 \rangle$ is the field with 4 elements.