1. Find the subgroup of $\mathbb{Z}$ generated by 4 and 6.

2. Find the subgroup of $\mathbb{R}^2$ generated by $(1,0)$ and $(0,1)$.

3. Consider $\sigma \in S_6$ defined using 2 line notation by

$$
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
3 & 5 & 6 & 4 & 2 & 1
\end{pmatrix}.
$$

Write out $\sigma$ explicitly as a product of transpositions and hence determine whether it is odd or even. Verify your answer by computing $\sigma \Delta$ where $\Delta$ is the difference product.

4. Show that $\Delta^2$ is a symmetric function.

5. Let $f(x_1,\ldots,x_n)$ be a complex polynomial. Show that the following two conditions on $f$ are equivalent: i) for any transposition $\tau$ we have $\tau.f = -f$ and ii) for any $\sigma \in S_n$ we have $\sigma.f = f$ when $\sigma$ is even and $\sigma.f = -f$ when $\sigma$ is odd. Such a polynomial is said to be anti-symmetric. Find some examples.

6. There is a right-handed version of all the results in lectures 6. For $H \leq G$, we define a right coset of $H$ in $G$ to be a set of the form $Hg := \{hg|h \in H\}$ for some $g \in G$. Show that $G$ is also a disjoint union of right cosets. (The sophisticated approach is via $G^{op}$). The set of right cosets is denoted $H \setminus G$. Let $\iota : G \longrightarrow G : g \mapsto g^{-1}$ be the inverse map. It is clearly a bijection. Show that $\iota(Hg) = g^{-1}H$ so $\iota$ gives a 1-1 correspondence between $H \setminus G$ and $G/H$. That is why there is no left or right index.

7. Let $G$ be the symmetric group on 4 symbols $S_4$ and $H$ be the subset $\{\sigma|\sigma(4) = 4\}$. Show that $H$ is a subgroup. Compute all the left and right cosets of $H$ in $G$. Verify Lagrange’s theorem and the 1-1 correspondence between left and right cosets given in question 5.

8. Let $H,K$ be subgroups of $G$ of order 3 and 5 respectively. Use Lagrange’s theorem to show that $H \cap K = 1$.

9. Let $G$ be a group with prime order. Use Lagrange’s theorem to find all subgroups of $G$. Show that $G$ is cyclic.

10. Using the previous exercise or otherwise, find all subgroups of $S_3$.

11. Show associativity of the subset product claimed in lecture 7 i.e. for subsets $K_1,K_2,K_3$ of a group $G$ we have $(K_1K_2)K_3 = K_1(K_2K_3)$. 

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(by Daniel Chan)
12. Let \( G = \mathbb{C}^\ast \) and \( H \) be the subset of complex numbers of modulus 1. Show that \( H \) is a normal subgroup of \( G \) and describe the cosets of \( H \). Show that \( G/H \) is isomorphic to a subgroup of \( \mathbb{R}^\ast \).

13. Show that \( A_n \leq S_n \) is generated by 3-cycles.

14. Let \( G = GL_2 \) and let \( H \) be the subgroup of elements of the form \( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \) where \( a, c \in \mathbb{R}^\ast \) and \( b \in \mathbb{R} \). Compute all the left and right cosets of \( H \) in \( G \). If you know some projective geometry you may wish to show that \( G/H \) can be naturally identified with the real projective line.

15. Compute explicitly all cosets of \( SL_n := \{ M \in GL_n \mid \det M = 1 \} \) in \( GL_n \).

16. Compute all cosets of \( O_2 \) in \( GL_2 \).

17. Let \( G \) be a group and \( H \) be a subgroup of index two. Show that \( H \) is normal.

18. Why is \( H = A_n \) normal in \( G = S_n \)? Find a group isomorphic to \( G/H \).

19. Let \( z \in \mathbb{C}^\ast \) and \( \phi \) be multiplication by \( z \). Is \( \phi \) a group homomorphism from a) \( \mathbb{C} \rightarrow \mathbb{C} \), b) \( \mathbb{C}^\ast \rightarrow \mathbb{C}^\ast \)?

20. Find all isomorphisms \( \phi : \mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z} \) where \( p \) is prime.

21. Isomorphic groups should be identical as far as their group structure is concerned. To illustrate this, consider an isomorphism \( \phi : G \rightarrow G' \).

   Show

   (a) \( G \) is abelian if and only if \( G' \) is.

   (b) \( G, G' \) have the same order.

   (c) There is a natural bijection between the the subgroups of \( G \) and the subgroups of \( G' \). It preserves orders, inclusions and normality.

   (d) If \( g \in G \) has order \( n \), so does \( \phi(g) \).

22. Show that \( S_3 \) and \( \mathbb{Z}/6\mathbb{Z} \) both have order 6 (so are isomorphic sets) but are not isomorphic as groups.

23. Fix an integer \( n \geq 2 \). Suppose that \( G \) is a group with distinct elements \( \{1, \sigma, \sigma^2, \ldots, \sigma^{n-1}, \tau, \sigma \tau, \ldots, \sigma^{n-1} \tau\} \) where \( \sigma^n = 1 = \tau^2 \) and \( \tau \sigma = \sigma^{-1} \tau \). Show that \( G \) is isomorphic to \( D_n \).

24. Find all normal subgroups \( H \) of \( D_n \). Show that \( G/H \) is dihedral or cyclic. (N.B. This means isomorphic to a dihedral group or cyclic group).

25. For \( \sigma \in S_n \) we let \( \Phi(\sigma) \) be the linear transformation \( \Phi(\sigma) : (x_1, \ldots, x_n)^t \mapsto (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})^t \). Show that \( \Phi : S_n \rightarrow GL_n \) is a group homomorphism. Determine its image.
26. Show that $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to the group $\mu_n$ introduced in problem sheet 1.

27. Let $f : S \rightarrow T$ be a bijection of sets. Show that $\phi : \text{Perm } S \rightarrow \text{Perm } T : \sigma \mapsto f\sigma f^{-1}$ is an isomorphism.

28. Let $W$ be a 2-dimensional subspace of $\mathbb{R}^3$. Recall that $\mathbb{R}^3, \mathbb{R} = \mathbb{R}^1$ are groups with group multiplication given by vector addition and that $W$ is a subgroup of $\mathbb{R}^3$. Show that $\mathbb{R}^3/W$ is isomorphic to $\mathbb{R}$ as a group. (In fact, there is a natural vector space structure on $\mathbb{R}^3/W$ and the isomorphism is even of vector spaces).