1. (50 marks total) The following are each worth 5 marks. Justify your answers with a brief explanation (but be careful to mention the key points).

   a) Are $3, 3i$ associates in $\mathbb{Z}[i]$?
   
   b) What is the order of the rotational symmetry group of a tetrahedron?
   
   c) Is $x^3 + 2x^2 - 2x + 6$ irreducible in $\mathbb{Q}[x]$? Is $\mathbb{Q}[x]/\langle x^3 + 2x^2 - 2x + 6 \rangle$ a field?
   
   d) Simplify $\mathbb{C}[x, y]/\langle y - x \rangle$. Is $\langle y - x \rangle \triangleleft \mathbb{C}[x, y]$ prime?
   
   e) Is $\mathbb{Z}[x]$ a UFD? Is it a PID?
   
   f) Is $[\mathbb{Q}(\cos \frac{\pi}{16}) : \mathbb{Q}]$ a power of two?
   
   g) Consider an algebraic field extension $K/E$ and a finite field extension $E/F$. Is $K/F$ algebraic?
   
   h) Let $S$ be a $G$-set and $g \in G$. Show $S^g = S^{g^{-1}}$.
   
   i) What are all the ideals of $\mathbb{R}[x]/\langle x^2 - x \rangle$?
   
   j) What is the group of units $(\mathbb{C}[x, y]/\langle xy - 1 \rangle)^*$?

2. (10 marks) In this question, we work in the ring $R = \mathbb{Z}[i\sqrt{2}]$. Find the greatest common divisor of $2i\sqrt{2}$ and $2 + i\sqrt{2}$ in $R$ (be sure to show working). What is the ideal $\langle 2i\sqrt{2}, 2 + i\sqrt{2}, 1 + 9i\sqrt{2} \rangle$? (Make sure your answer is in simplest form!)

3. (10 marks) In each question below, make sure you justify your answer fully. Let

   \[ \alpha := \sqrt[3]{2 + \sqrt{2}}. \]

   a) Is $\alpha$ algebraic over $\mathbb{Q}$? If so, find the minimal polynomial of $\alpha$ over $\mathbb{Q}$.
   
   b) What is $[\mathbb{Q}(\alpha) : \mathbb{Q}]$?
   
   c) What is the minimal polynomial of $\sqrt[3]{2}$ over $\mathbb{Q}$?
   
   d) Is $\sqrt[3]{2} \in \mathbb{Q}(\alpha)$?

4. (8 marks) Consider the subset $S := \{ p(x) \in \mathbb{R}[x] | p(-x) = p(x) \}$ of even polynomials.

   a) Show that $S$ is a subring of $\mathbb{R}[x]$.
   
   b) Consider the map $\phi : \mathbb{R}[y] \rightarrow \mathbb{R}[x] : p(y) \mapsto p(x^2)$. Is $\phi$ an homomorphism? Justify your answer with a brief explanation.

Please see over . . .
c) Show that the image of $\phi$ is $S$.

d) Is $\mathbb{R}[y] \simeq S$? Justify fully, your answer.

5. (10 marks) In this question, we let $R = \mathbb{Z}[i]$.

a) Show that $1 + i$ is irreducible in $R$.

b) Show that $2$ is reducible in $R$.

c) Factorise $2x^2 - 18$ into irreducibles in $R[x]$. Make sure you prove that your factors are indeed irreducible.

6. (12 marks) Let $G$ be the group of $3 \times 3$-matrices below

$$G = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \mid a, b, c \in \{1, -1\} \right\}.$$ 

We let $G$ act on $\mathbb{R}^3$ by matrix multiplication, that is, if $v \in \mathbb{R}^3$ then $g.v := gv$.
Consider also the following vectors in $\mathbb{R}^3$

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

a) What is the order of $G$?

b) Find the $G$-orbit of $v_1$ and its stabiliser.

c) Prove that $G.v_1 \simeq G.v_2$ as $G$-sets.

d) Prove that $G.v_1$ is not isomorphic to $G.v_3$ as $G$-sets.