MATH3710: Higher Algebra I, Problem Sheet 6

1. Describe all abelian groups of order 24 up to isomorphism.

2. Find the $n$-torsion subgroup $A_n$ of $A = \mathbb{Q}/\mathbb{Z}$ and $A = \mathbb{Z}/m\mathbb{Z}$.

3. Let $G = \mathbb{Z}^3$ and $H$ be the subgroup generated by

$$\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ -3 \end{pmatrix}.$$ 

Write $G/H$ as a product of cyclic groups. Hint: Use the proof of the classification of finitely generated abelian groups given in lecture 22.

4. Let $G = \mathbb{Z}/h_1\mathbb{Z} \times \ldots \times \mathbb{Z}/h_r\mathbb{Z} \times \mathbb{Z}^s$. What is the torsion subgroup $H$ and $G/H$. Show that the number $s$ depends only on the isomorphism class of $G$.

5. Find a composition series for $S_4$.

6. For a finite group $G$, define the $k$-th derived group inductively by, $G^{(k)} := G^{(k-1)}$ and $G^{(1)} = G'$. Show that

$$1 \leq G^{(k)} \leq G^{(k-1)} \leq \ldots \leq G' \leq G$$

is a normal chain of subgroups of $G$. Show that $G$ is solvable if and only if $G^{(r)} = 1$ for some $r \in \mathbb{N}$.

7. Is the dihedral group solvable? What is the length of a dihedral group?