MATH3710: Higher Algebra I, Problem Sheet 3

1. Recall that $\mathbb{R}$ is a group when endowed with addition and $H := \{z \in \mathbb{C}||z| = 1\}$ is a subgroup of $\mathbb{C}^*$. Using the exponential function and the first isomorphism theorem, show that $H$ is isomorphic to a quotient group of $\mathbb{R}$.

2. Let $\phi : \mathbb{C}^* \rightarrow \mathbb{C}^* : z \mapsto z^n$ for some positive integer $n$. Show that $\phi$ is a group homomorphism. Find $\ker \phi$, $\im \phi$. What isomorphism does the first isomorphism theorem give? Verify that the fibres of $\phi$ are indeed the cosets of $\ker \phi$.

3. Weak version of Chinese remainder theorem. Let $m, n$ be relatively prime positive integers. Consider the homomorphism $\phi : \mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ defined by $\phi(a) = (a + m\mathbb{Z}, a + n\mathbb{Z})$. Find $\ker \phi$. Compare the orders of $\mathbb{Z}/\ker \phi$ and $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ to determine the image of $\phi$. Use the first isomorphism theorem to find which cyclic group $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ is isomorphic to.

4. Let $T, U$ be sets and $S$ be their union. Consider the subset $G$ of $\text{Perm} S$ consisting of permutations $\sigma$ such that $\sigma(T) = T, \sigma(U) = U$ (Note that $G$ is a subgroup). Use the universal property of products to construct a group isomorphism $G \sim \text{Perm} T \times \text{Perm} U$.

5. Let $G$ be the dihedral group of order $2n$ and $N$ the (unique) cyclic subgroup of order $n$ ($N = \langle \sigma \rangle$ in the lecture notes). Let $H$ be the group generated by any $\tau \notin N$. Verify the third isomorphism theorem in this case and compute explicitly the isomorphism.

6. Suppose $N \trianglelefteq G, N' \trianglelefteq G'$. Show that $N \times N'$ is naturally a normal subgroup of $G \times G'$ and show $(G \times G')/(N \times N') \simeq (G/N) \times (G'/N')$.