Asymptotic enumeration of sparse uniform hypergraphs with given degrees

Vladimir Blinovsky∗
Instituto de Matemática e Estatística
Universidade de São Paulo, 05508-090, Brazil
Institute for Information Transmission Problems
Russian Academy of Sciences
Moscow 127994, Russia
vblinovs@yandex.ru

Catherine Greenhill†
School of Mathematics and Statistics
The University of New South Wales
Sydney NSW 2052, Australia
c.greenhill@unsw.edu.au

9 June 2015

Abstract

Let \( r \geq 2 \) be a fixed integer. For infinitely many \( n \), let \( k = (k_1, \ldots, k_n) \) be a vector of nonnegative integers such that their sum \( M \) is divisible by \( r \). We present an asymptotic enumeration formula for simple \( r \)-uniform hypergraphs with degree sequence \( k \). (Here “simple” means that all edges are distinct and no edge contains a repeated vertex.) Our formula holds whenever the maximum degree \( k_{\text{max}} \) satisfies \( k_{\text{max}}^3 = o(M) \).

1 Introduction

Hypergraphs are combinatorial structures which can model very general relational systems, including some real-world networks [3, 4, 6]. Formally, a hypergraph or a set system is defined as a pair \((V, E)\), where \( V \) is a finite set and \( E \) is a multiset of multisubsets of \( V \). (We refer to elements of \( E \) as edges.) Note that under this definition, a hypergraph may contain repeated edges and an edge may contain repeated vertices.

∗Supported by FAPESP (2012/13341-8, 2013/07699-0) and NUMEC/USP (Project MaCLinC/USP).
†Supported by the Australian Research Council grants DP120100197 and DP140101519.
If a vertex \( v \) has multiplicity at least 2 in the edge \( e \), we say that \( v \) is a loop in \( e \). A hypergraph is simple if it has no loops and no repeated edges. Here it is possible that distinct edges may have more than one vertex in common. Let \( r \geq 2 \) be a fixed integer. We say that the hypergraph \( (V,E) \) is \( r \)-uniform if each edge \( e \in E \) contains exactly \( r \) vertices (counting multiplicities). Uniform hypergraphs are a particular focus of study, not least because a 2-uniform hypergraph is precisely a graph. We seek an asymptotic enumeration formula for the number of \( r \)-uniform simple hypergraphs with a given degree sequence, when \( r \geq 3 \) is constant and the maximum degree is not too large (the sparse range).

To state our result precisely, we need some definitions. Let \( k_{i,n} \) be a nonnegative integer for all pairs \((i,n)\) of integers which satisfy \( 1 \leq i \leq n \). Then for each \( n \geq 1 \), let \( k_n = k(n) = (k_{1,n}, \ldots, k_{n,n}) \). We usually write \( k_i \) instead of \( k_{i,n} \). Define \( M = \sum_{i=1}^{n} k_i \). We assume that \( M \) is divisible by \( r \) for an infinite number of values of \( n \), and tacitly restrict ourselves to such \( n \).

We write \( (a)_m \) to denote the falling factorial \( a(a-1) \cdots (a-m+1) \), for integers \( a \) and \( m \). For each positive integer \( t \), let \( M_t = \sum_{i=1}^{n} (k_t)_i \). Notice that \( M_1 = M \) and that \( M_t \leq k_{\max} M_{t-1} \) for \( t \geq 2 \).

Let \( \mathcal{H}_r(k) \) be the set of simple \( r \)-uniform hypergraphs on the vertex set \( \{1,2,\ldots,n\} \) with degrees given by \( k = (k_1,\ldots,k_n) \). Our main theorem is the following.

**Theorem 1.1.** Let \( r \geq 3 \) be a fixed integer. Suppose that \( n \rightarrow \infty \), \( M \rightarrow \infty \) and that \( k_{\max} \) satisfies \( k_{\max} \geq 2 \) and \( k_{\max}^3 = o(M) \). Then

\[
|\mathcal{H}_r(k)| = \frac{M!}{(M/r)! (r!)^{M/r} \prod_{i=1}^{n} k_i!} \exp \left( -\frac{(r-1) M_2}{2M} + O\left(\frac{k_{\max}^3}{M}\right) \right).
\]

As a corollary, we immediately obtain the corresponding formula for regular hypergraphs. Let \( \mathcal{H}_r(k,n) \) denote the set of all \( k \)-regular \( r \)-uniform hypergraphs on the vertex set \( \{1,\ldots,n\} \), where \( k \geq 2 \) is an integer, which may be a function of \( n \).

**Corollary 1.2.** Suppose that \( n \rightarrow \infty \) and that \( k \) satisfies \( k \geq 2 \) and \( k^2 = o(n) \). Then

\[
|\mathcal{H}_r(k,n)| = \frac{(kn)!}{(kn/r)! (r!)^{kn/r} (k!)^n} \exp \left( -\frac{1}{2} \frac{k-1}{r} (r-1) + O\left(\frac{k^2}{n}\right) \right).
\]

1.1 History

In the case of graphs, the best asymptotic formula in the sparse range is given by McKay and Wormald [11]. See that paper for further history of the problem. Note that their formula
has a similar form to ours, but with many more terms in the exponential factor. This is due to the fact that it is harder to avoid creating a repeated edge with a switching when \( r = 2 \).

The dense range for \( r = 2 \) was treated in \([9, 10]\), but there is a gap between these two ranges in which nothing is known.

An early result in the asymptotic enumeration of hypergraphs was given by Cooper et al. \([1]\), who considered simple \( k \)-regular hypergraphs when \( k = O(1) \). Dudek et al. \([2]\) proved an asymptotic formula for the number of simple \( k \)-regular hypergraphs graphs with \( k = o(n^{1/2}) \). A restatement of their result in our notation is the following:

**Theorem 1.3.** (\([2, \text{Theorem 1}]\)) For each integer \( r \geq 3 \), define

\[
\kappa = \kappa(r) = \begin{cases} 
1 & \text{if } r \geq 4, \\
\frac{1}{2} & \text{if } r = 3.
\end{cases}
\]

Let \( \mathcal{H}(r, k) \) denote the set of all simple \( k \)-regular \( r \)-uniform hypergraphs on the vertex set \( \{1, \ldots, n\} \). For every \( r \geq 3 \), if \( k = o(n^{\kappa}) \) then

\[
|\mathcal{H}(r, k)| = \frac{(kn)!}{(kn/r)! (r)!^{kn/r} (k!)^n} \exp \left(-\frac{1}{2} (k - 1)(r - 1)(1 + O(\delta(n)))\right)
\]

where \( \delta(n) = (kn)^{-1/2} + k/n \).

Note that the factor outside the exponential part matches ours (see Corollary 1.2), and that the exponential part of their formula can be rewritten as

\[
\exp \left(-\frac{1}{2} (k - 1)(r - 1) + O(k\delta(n))\right)
\]

with relative error

\[
O(k\delta(n)) = O\left(\sqrt{k/n} + k^2/n\right).
\]

This relative error is only \( o(1) \) when \( k^2 = o(n) \), matching the range of \( k \) covered by Corollary 1.2. Hence Theorem 1.1 can be seen as an extension of \([2]\) to irregular degree sequences.

For an asymptotic formula for the number of dense simple \( r \)-uniform hypergraphs with a given degree sequence, see \([7]\).

### 1.2 The model, some early results and a plan of the proof

We work in a generalisation of the configuration model. Let \( B_1, B_2, \ldots, B_n \) be disjoint sets, which we call *cells*, and define \( \mathcal{B} = \bigcup_{i=0}^n B_i \). Elements of \( \mathcal{B} \) are called points. Assume that
cell $B_i$ contains exactly $k_i$ points, for $i = 1, \ldots, n$. We assume that there is a fixed ordering on the $M$ points of $\mathcal{B}$.

Denote by $\Lambda_r(k)$ the set of all unordered partitions $Q = \{U_1, \ldots, U_{M/r}\}$ of $\mathcal{B}$ into $M/r$ parts, where each part has exactly $r$ points. Then

$$|\Lambda_r(k)| = \frac{M!}{(M/r)! (r!)^{M/r}}. \quad (1.1)$$

Each partition $Q \in \Lambda_r(k)$ defines a hypergraph $G(Q)$ on the vertex set $\{1, \ldots, n\}$ in a natural way: vertex $i$ corresponds to the cell $B_i$, and each part $U \in Q$ gives rise to an edge $e_U$ such that the multiplicity of vertex $i$ in $e_U$ equals $|U \cap B_i|$, for $i = 1, \ldots, n$. Then $G(Q)$ is an $r$-uniform hypergraph with degree sequence $k$. The partition $Q \in \Lambda_r(k)$ is called simple if $G(Q)$ is simple.

The edge $e_U$ has a loop at $i$ if and only if $|U \cap B_i| \geq 2$. In this case, each pair of distinct points in $U \cap B_i$ is called a loop in $U$. We reserve the letters $e, f$ for edges in a hypergraph, and use $U, W$ for parts in a partition $Q$ (that is, in the configuration model).

Now we will consider random partitions. Each hypergraph in $\mathcal{H}_r(k)$ corresponds to exactly

$$\prod_{i=1}^{n} k_i!$$

partitions $Q \in \Lambda_r(k)$. Hence, when $Q \in \Lambda_r(k)$ is chosen uniformly at random, conditioned on $G(Q)$ being simple, the probability distribution of $G(Q)$ is uniform over $\mathcal{H}_r(k)$. Let $P_r(k)$ denote the probability that a partition $Q \in \Lambda_r(k)$ chosen uniformly at random is simple. Then

$$|\mathcal{H}_r(k)| = \frac{M!}{(M/r)! (r!)^{M/r}} \prod_{i=1}^{n} k_i! \cdot P_r(k). \quad (1.2)$$

Hence it suffices to show that $P_r(k)$ equals the exponential factor in the statement of Theorem 1.1. As a first step, we identify several events which have probability $O(k_3^{\max}/M)$ in the uniform probability space over $\Lambda_r(k)$.

The following lemma will be used repeatedly. In most applications, $c$ will be a small positive integer. (Throughout the paper, “log” denotes the natural logarithm.)

**Lemma 1.4.** Let $U_1, \ldots, U_c$ be fixed, disjoint $r$-subsets of the set of points $\mathcal{B}$, where $r \geq 3$ is a fixed integer and $c = o(M^{1/2})$. The probability that a uniformly random $Q \in \Lambda_r(k)$ contains the parts $\{U_1, \ldots, U_c\}$ is

$$(1 + o(1)) \frac{(r-1)!^c}{M^{c(r-1)}}.$$
Proof. Using (1.1), the required probability is
\[
\frac{r!^c (M/r)^c}{(M)^{rc}} = \frac{(r-1)!^c}{M^{(r-1)c}} \exp \left( - \sum_{j=0}^{rc-1} \log(1 - j/M) + \sum_{i=0}^{c-1} \log(1 - ri/M) \right) \\
= \frac{(r-1)!^c}{M^{(r-1)c}} \exp \left( O \left( \frac{r^2c^2}{M} \right) \right).
\]
But \( r^2c^2 = o(M) \) by assumption, which completes the proof. \( \square \)

Let
\[N = \max\{\lceil \log M \rceil, \lceil 9(r-1)M_2/M \rceil\} .\]
Now define \( \Lambda_r^+(k) \) to be the set of partitions \( Q \in \Lambda_r(k) \) which satisfy the following properties:

(i) For each part \( U \in Q \) we have \(|U \cap B_i| \leq 2 \) for \( i = 1, \ldots, n \).

(ii) For each part \( U \in Q \) there is at most one \( i \in \{1, \ldots, n\} \) with \(|U \cap B_i| = 2\).

(iii) For each pair \((U_1, U_2)\) of distinct parts in \( Q \), the intersection \( e_1 \cap e_2 \) of the corresponding edges contains at most 2 vertices. (It is possible that \( e_1 \cap e_2 \) consists of a loop.)

(iv) There are at most \( N \) parts which contain loops.

Note in particular that whenever \( r \geq 3 \), property (iii) implies that \( G(Q) \) has no repeated edges.

**Lemma 1.5.** Under the assumptions of Theorem 1.1, we have
\[
\frac{|\Lambda_r^+(k)|}{|\Lambda_r(k)|} = 1 + O(k_\max^3/M).
\]

*Proof.* Consider \( Q \in \Lambda_r(k) \) chosen uniformly at random.

(i) The expected number of parts in \( Q \) which contain three or more points from the same cell is
\[
O \left( \frac{M_3M^{r-3}}{M^{r-1}} \right) = O(k_\max^2/M) ,
\]
using Lemma 1.4. Hence, the probability that property (i) fails to hold is also \( O(k_\max^2/M) \).

(ii) Similarly, the expected number of parts in \( Q \) which contain two loops (where each loop is from a distinct cell) is
\[
O \left( \frac{M_2^2M^{r-4}}{M^{r-1}} \right) = O(k_\max^2/M) .
\]
(iii) Using Lemma 1.4, the expected number of ordered pairs of distinct parts \((U_1, U_2)\) which give rise to edges \(e_1, e_2\) such that \(|e_1 \cap e_2| \geq 3\) is

\[
O\left(\frac{M_2^3 M^{2(r-3)} + M_2 M_4 M^{2(r-3)}}{M^{2(r-1)}}\right) = O(k_{\text{max}}^3/M).
\]

(Here the first term arises if \(e_1 \cap e_2\) does not contain a loop while the second term covers the possibility that \(e_1 \cap e_2\) contains a loop. By (i) we can assume that \(e_1 \cap e_2\) contains at least two distinct vertices.)

(iv) Let \(\ell = N + 1\). We bound the expected number of sets \(\{U_1, \ldots, U_\ell\}\) of \(\ell\) parts which each contain a loop. Given \((U_1, \ldots, U_{\ell-1})\), there are at most \(M_2 M^{r-2}/(2(r-2)!}\) choices for \(U_{\ell}\). Hence there are

\[
O\left(\frac{1}{\ell!} \left(\frac{M_2 M^{r-2}}{2(r-2)!}\right)^\ell\right)
\]
possible sets \(\{U_1, \ldots, U_\ell\}\) of parts which each contain a loop. Now

\[
\ell = O(N) = O(k_{\text{max}} + \log M) = o(M^{1/2}),
\]
by definition of \(N\). Hence Lemma 1.4 applies, and we conclude that the expected number of sets of \(\ell = N + 1\) parts which each contain a loop is

\[
O\left(\frac{1}{\ell!} \left(\frac{(r-1)M_2}{2M}\right)^\ell\right) = O\left(\left(\frac{e(r-1)M_2}{2\ell M}\right)^\ell\right) = O\left(\frac{((e/18)^{\log M})}{1/M}\right) = o(1/M),
\]
completing the proof.

In Section 2 we will calculate \(|\Lambda^+_r(k)|\) by analysing switchings which make local changes to a partition to reduce (or increase) the number of loops by precisely 1.

\section{The switchings}

For a given nonnegative integer \(\ell\), let \(C_\ell\) be the set of partitions \(Q \in \Lambda^+_r(k)\) with exactly \(\ell\) parts which contain a loop. Then partitions in \(C_0\) give rise to hypergraphs in \(\mathcal{H}_r(k)\). Now \(C_0\) is nonempty whenever \(r\) divides \(M\), and we restrict ourselves to this situation. Hence it follows from Lemma 1.5 that

\[
\frac{1}{P_r(k)} = \left(1 + O(k_{\text{max}}^3/M)\right) \sum_{\ell=0}^N \frac{|C_\ell|}{|C_0|}.
\]  

(2.1)

We estimate the above sum using a switching designed to remove loops.
An $\ell$-switching in a partition $Q$ is specified by a 4-tuple $(x_1, x_2, y_1, y_2)$ of points where $x_1$ belongs to the part $U$, and $y_j$ belongs to the part $W_j$ for $j = 1, 2$, such that:

- $U$, $W_1$ and $W_2$ are distinct parts of $Q$,
- $y_1$ and $y_2$ belong to distinct cells, and
- $U$ contains a loop $\{x_1, x_2\}$ (so in particular, $x_1$ and $x_2$ belong to the same cell).

The $\ell$-switching maps $Q$ to the partition $Q'$ defined by

$$Q' = (Q - \{U, W_1, W_2\}) \cup \{\hat{U}, \hat{W}_1, \hat{W}_2\}$$

where

$$\hat{U} = (U - \{x_1, x_2\}) \cup \{y_1, y_2\}, \quad \hat{W}_1 = (W_1 - \{y_1\}) \cup \{x_1\}, \quad \hat{W}_2 = (W_2 - \{y_2\}) \cup \{x_2\}.$$  

This operation is illustrated in Figure 1. It is the same operation used by Dudek et al. [2], but we use a somewhat different approach when analysing the switching.

![Diagram of an \ell-switching]

Let $e$ be the edge of $G(Q)$ corresponding to $U$, and let $f_j$ be the edge of $G(Q)$ corresponding to $W_j$, for $j = 1, 2$. Similarly, let $\hat{e}$ be the edge of $G(Q')$ corresponding to $\hat{U}$, and let $\hat{f}_j$ be the edge of $G(Q')$ corresponding to $\hat{W}_j$ for $j = 1, 2$.

Given $Q \in \mathcal{C}_\ell$, we say that the $\ell$-switching specified by the 4-tuple of points $(x_1, x_2, y_1, y_2)$ is legal for $Q$ if the resulting partition $Q'$ belongs to $\mathcal{C}_{\ell-1}$, and otherwise we say that the switching is illegal for $Q$. 

7
Lemma 2.1. With notation as above, if the \( \ell \)-switching \((x_1, x_2, y_1, y_2)\) is illegal for \(Q\) then at least one of the following conditions must hold:

(I) At least one of \(W_1, W_2\) contains a loop.

(II) \(e, f_1\) and \(f_2\) are not pairwise disjoint.

(III) Some edge of \(G(Q) \setminus \{e, f_1, f_2\}\) intersects both \(e\) and \(f_j\), for some \(j \in \{1, 2\}\).

Proof. Given \(Q \in \mathcal{C}_\ell\), suppose that the 4-tuple \((x_1, x_2, y_1, y_2)\) specifies an \(\ell\)-switching in \(Q\) such that the resulting partition \(Q'\) does not belong to \(\mathcal{C}_{\ell-1}\).

It could be that \(Q' \in \Lambda_\ell^+(k)\) but that \(Q'\) has strictly more than \(\ell - 1\) parts which contain a loop. Here the \(\ell\)-switching has (accidently) introduced at least one new loop. But this implies that (II) holds, since we know that \(y_1\) and \(y_2\) do not belong to the same cell.

Next, suppose that \(Q' \in \Lambda_\ell^+(k)\) but that \(Q'\) has at most \(\ell - 2\) parts which contain a loop. This means that the \(\ell\)-switching has removed more than one loop. Then property (I) must hold: the point \(y_j\) must have been involved in a loop in \(W_j\) for some \(j \in \{1, 2\}\).

It remains to consider the case that \(Q' \not\in \Lambda_\ell^+(k)\). Then at least one of the properties (i)–(iv) used to define \(\Lambda_\ell^+(k)\) no longer holds for \(Q'\). Arguing as above, if (i), (ii) or (iv) fails then we have introduced at least one loop, or increased the multiplicity of a vertex in some edge from 2 to at least 3. This implies that (I) or (II) holds, using arguments similar to those above.

Finally, suppose that (iii) fails for \(Q'\). Then \(G(Q')\) has a pair of edges which intersect in at least 3 vertices. We say that this pair of edges has large intersection. At least one of the new edges \(\hat{e}, \hat{f}_1, \hat{f}_2\) must be involved in any such pair, since \(Q \in \Lambda_\ell^+(k)\).

If \(\hat{f}_1\) and \(\hat{f}_2\) have large intersection then \(f_1\) and \(f_2\) are not disjoint, which shows that (II) holds. Similarly, if \(\hat{e}\) and \(\hat{f}_j\) have large intersection for some \(j \in \{1, 2\}\) then \(e\) and \(f_j\) are not disjoint, and (II) holds. Now suppose that an edge \(e' \in G(Q') \setminus \{\hat{e}, \hat{f}_1, \hat{f}_2\}\) has large intersection with one of the new edges. Note that \(e'\) is also an edge of \(G(Q) \setminus \{e, f_1, f_2\}\).

- If \(e'\) has large intersection with \(\hat{f}_j\) for some \(j \in \{1, 2\}\) then \(e'\) must contain the vertex corresponding to the point \(x_j\), or else \(e'\) and \(f_j\) would have large intersection in \(G(Q)\), contradicting the fact that \(Q \in \Lambda_\ell^+(k)\). Furthermore, \(e' \cap \hat{f}_j\) contains at least one other vertex, corresponding to a point in \(\hat{W}_j \setminus \{x_j\} = W_j \setminus \{y_j\}\). Hence \(e'\) intersects both \(e\) and \(f_j\) in \(G(Q)\), showing that (III) holds.

- If \(e'\) has large intersection with \(\hat{e}\) then \(e'\) must contain the vertex corresponding to \(y_j\) for some \(j \in \{1, 2\}\) (perhaps both), otherwise \(e'\) and \(e\) would have large intersection in
G(Q), a contradiction. Even if e' contains both of these vertices, it must still contain a vertex corresponding to a point in \( \hat{U} \setminus \{y_1, y_2\} = U \setminus \{x_1, x_2\} \). Hence e' intersects both \( f_j \) and e in \( G(Q) \) for some \( j \in \{1, 2\} \), which again proves that (III) holds.

This completes the proof. \( \square \)

A reverse \( \ell \)-switching in a given partition \( Q' \) is the reverse of an \( \ell \)-switching. It is described by a 4-tuple \( (x_1, x_2, y_1, y_2) \) of points, where \( \hat{W}_j \) is the part of \( Q' \) containing \( x_j \), for \( j = 1, 2 \), and \( y_1, y_2 \) are distinct points in the part \( \hat{U} \) of \( Q' \), such that

- \( \hat{U}, \hat{W}_1 \) and \( \hat{W}_2 \) are distinct parts of \( Q' \),
- \( x_1 \) and \( x_2 \) belong to the same cell, and
- \( y_1 \) and \( y_2 \) belong to distinct cells.

This reverse \( \ell \)-switching acting on \( Q' \) produces the partition \( Q \) defined by (2.2), as depicted in Figure 1 by following the arrow in reverse. Given \( Q' \in C_{\ell-1} \), we say that the reverse \( \ell \)-switching specified by \( (x_1, x_2, y_1, y_2) \) is legal for \( Q' \) if the resulting partition \( Q \) belongs to \( C_\ell \), and otherwise we say that the switching is illegal for \( Q' \). For completeness we give the full proof of the following, though it is very similar to the proof of Lemma 2.1.

**Lemma 2.2.** With notation as above, if the reverse \( \ell \)-switching specified by \( (x_1, x_2, y_1, y_2) \) is illegal for \( Q' \in C_{\ell-1} \) then at least one of the following conditions must hold:

(1') At least one of \( \hat{U}, \hat{W}_1, \hat{W}_2 \) contains a loop.

(1')' \( \hat{e} \cap \hat{f}_j \neq \emptyset \) for some \( j \in \{1, 2\} \).

(III') Some edge of \( G(Q') \setminus \{\hat{e}, \hat{f}_1, \hat{f}_2\} \) intersects both \( \hat{e} \) and \( \hat{f}_j \) for some \( j \in \{1, 2\} \).

**Proof.** Fix \( Q' \in C_{\ell-1} \) and let \( (x_1, x_2, y_1, y_2) \) describe an reverse \( \ell \)-switching such that the resulting partition \( Q \) does not belong to \( C_\ell \).

If \( Q \in \Lambda^+_r(k) \) but \( Q \) has more than \( \ell \) parts which contain loops then an extra loop has been unintentionally introduced. In this case, either \( \hat{W}_j \setminus \{x_j\} \) contains a point from the same cell as \( y_j \), or \( \hat{U} \setminus \{y_1, y_2\} \) contains a point from the same cell as \( x_j \), for some \( j \in \{1, 2\} \). In either case we have \( \hat{e} \cap \hat{f}_j \neq \emptyset \), so (II') holds. Next, suppose that \( Q \in \Lambda^+_r(k) \) but that \( Q \) has at most \( \ell - 1 \) parts which contain a loop. Then the reverse switching has removed at least one loop, which implies that (I') holds.
Now suppose that \( Q \not\in \Lambda^+_{\ell}(k) \). Then one of the properties (i)–(iv) fail for \( Q \). If (i), (ii) or (iv) fail then arguing as above we see that (I') or (II') holds. Now suppose that (iii) fails. Then some edge of \( G(Q) \) has large intersection with one of \( e, f_1, f_2 \) (recalling that terminology from the proof of Lemma 2.1). Now \( f_1 \) and \( f_2 \) cannot have large intersection, since their intersection is contained in the intersection of \( \hat{f}_1 \) and \( \hat{f}_2 \), and \( Q' \in \Lambda^+_{\ell}(k) \). If \( e \) and \( f_j \) have large intersection for some \( j \in \{1, 2\} \) then either this intersection contains the vertex corresponding to \( x_j \) (and hence \( \hat{W}_j \) contains a loop), or the intersection contains the vertex corresponding to \( y_j \) (and hence \( \hat{U} \) contains a loop), or \( \hat{e} \cap \hat{f}_j \neq \emptyset \). Again (I') or (II') hold.

Finally, suppose that the large intersection involves an edge \( e' \in G(Q) \setminus \{e, f_1, f_2\} \). Then \( e' \) also belongs to \( G(Q') \setminus \{\hat{e}, \hat{f}_1, \hat{f}_2\} \). If \( e' \) has large intersection with \( e \) in \( G(Q) \) then \( e' \) contains the vertex corresponding to the point \( x_j \), for some \( j \in \{1, 2\} \) (or else \( e' \) and \( \hat{e} \) have large overlap in \( G(Q') \), a contradiction), and \( e' \) contains at least one vertex corresponding to a point of \( U \setminus \{x_1, x_2\} = \hat{U} \setminus \{y_1, y_2\} \). Therefore \( e' \) overlaps both \( \hat{e} \) and \( \hat{f}_j \), so (III') holds. Similarly, if \( e' \) has large intersection with \( \hat{f}_j \) for some \( j \in \{1, 2\} \) then \( e' \) contains the vertex corresponding to \( y_j \) (or else \( e' \cap \hat{f}_j \) is large in \( G(Q') \), a contradiction), and \( e' \) contains at least one vertex corresponding to a point in \( W_j \setminus \{y_j\} = \hat{W}_j \setminus \{x_j\} \). Again, \( e' \) overlaps both \( \hat{e} \) and \( \hat{f}_j \), proving that (III') holds, as required. \( \square \)

Next we analyse these switchings to find a relationship between the sizes of \( C_\ell \) and \( C_{\ell - 1} \).

**Lemma 2.3.** Assume that the conditions of Theorem 1.1 hold and let \( \ell' \) be the first value of \( \ell \leq N \) such that \( C_\ell = \emptyset \), or \( \ell' = N + 1 \) if no such value exists. Then

\[
|C_\ell| = |C_{\ell - 1}| \frac{(r - 1)M_2}{2\ell M} \left(1 + O\left(\frac{k_{\max}^3 + \ell k_{\max}}{M_2}\right)\right)
\]

uniformly for \( 1 \leq \ell < \ell' \).

**Proof.** Fix \( \ell \in \{1, \ldots, \ell' - 1\} \) and let \( Q \in C_\ell \) be given. Define the set \( S \) of all 4-tuples \( (x_1, x_2, y_1, y_2) \) of distinct points such that

- \( y_1 \) and \( y_2 \) belong to distinct cells,

- \( \{x_1, x_2\} \) is a loop in \( U \) and \( y_j \in W_j \) for \( j = 1, 2 \), for some distinct parts \( U, W_1, W_2 \in Q \), and

- neither \( W_1 \) nor \( W_2 \) contain a loop.
Note that $S$ contains every 4-tuple which defines a legal $\ell$-switching from $Q$, so $|S|$ is an upper bound for the number of legal $\ell$-switchings which can be performed in $Q$.

There are precisely $2\ell$ ways to choose a pair of points $(x_1, x_2)$ which form a loop in some part $U$, using properties (i) and (ii) of the definition of $\Lambda^+_r(k)$. For an easy upper bound, there are at most $M^2$ ways to select $(y_1, y_2)$ with the required properties, giving $|S| \leq 2\ell M^2$. In fact

$$|S| = 2\ell M^2 \left( 1 + O \left( \frac{k_{\text{max}} + \ell}{M} \right) \right),$$

(2.3)

since there are precisely $M - r\ell$ ways to select a point $y_1$ which belongs to some part $W_1$ which does not contain a loop, and then there are $M - r(\ell + 1) + O(k_{\text{max}}) = M + O(k_{\text{max}} + \ell)$ ways to select a point $y_2$ which lies in a part $W_2$ which contains no loops and which is distinct from $W_1$, such that $y_1$ and $y_2$ not in the same cell.

We now find an upper bound for the number of 4-tuples in $S$ which give rise to illegal $\ell$-switchings, and subtract this value from $|S|$. By Lemma 2.1 it suffices to find an upper bound for the number of 4-tuples in $S$ which satisfy one of Conditions (I), (II), (III). First note that no 4-tuple in $S$ satisfies Condition (I), by definition of $S$.

If Condition (II) holds then $f_1 \cap f_2 \neq \emptyset$ or $e \cap f_j \neq \emptyset$ for some $j \in \{1, 2\}$. This occurs for at most $O(\ell k_{\text{max}} M)$ 4-tuples in $S$.

If Condition (III) holds then some edge $e'$ of $G(Q) \setminus \{e, f_1, f_2\}$ intersects two of $e$, $f_1$ and $f_2$. There are $O(\ell k_{\text{max}}^2 M)$ choices of 4-tuples in $S$ which satisfy this condition.

Combining these contributions, we find that there are

$$2\ell M^2 \left( 1 + O \left( \frac{k_{\text{max}}^2 + \ell}{M} \right) \right),$$

(2.4)

4-tuples $(x_1, x_2, y_1, y_2)$ which give a legal $\ell$-switching from $Q$.

Next, suppose that $Q' \in \mathcal{C}_{\ell-1}$ (and note that $\mathcal{C}_{\ell-1}$ is nonempty, by definition of $\ell'$). Let $S'$ be the set of all 4-tuples $(x_1, x_2, y_1, y_2)$ of distinct points such that

- $x_1$ and $x_2$ belong to the same cell,
- $x_j \in \widehat{W}_j$ for $j = 1, 2$ and and $y_1, y_2 \in \widehat{U}$, for some distinct parts $\widehat{U}, \widehat{W}_1, \widehat{W}_2$ of $Q'$, and
- $\widehat{U}$ does not contain a loop (so in particular, $y_1$ and $y_2$ belong to distinct cells).

Again, $S'$ contains every 4-tuple which describes a legal reverse $\ell$-switching from $Q'$, so the number of legal reverse $\ell$-switchings which may be performed in $Q'$ is at most $|S'|$. There are $M_2$ choices for $(x_1, x_2)$, and each such choice determines two distinct parts $\widehat{W}_1, \widehat{W}_2$ unless
\{x_1, x_2\} is a loop in some part of \( Q' \). Using properties (i) and (ii) of the definition of \( \Lambda_+^c(k) \), there are exactly \( 2(\ell - 1) \) choices of \( (x_1, x_2) \) such that \( \{x_1, x_2\} \) is a loop in \( Q' \). Next, there are precisely \( M - r(\ell - 1) \) choices for \( y_1 \) belonging to some part \( \tilde{U} \) which does not contain a loop, and then there are \( r - 1 \) choices for \( y_2 \in \tilde{U} \setminus \{y_1\} \). For a lower bound, there are at least \( (r - 1)(M - r(\ell + 1)) \) choices for \( (y_1, y_2) \) which ensure that \( \tilde{U} \) contains no loop and is distinct from both \( \tilde{W}_1 \) and \( \tilde{W}_2 \). Therefore

\[
(r - 1)(M - r(\ell + 1)) \left( M_2 - 2(\ell - 1) \right) \leq |S'| \leq (r - 1)(M - r(\ell - 1)) M_2,
\]

which implies that \( |S'| = (r - 1)M M_2 (1 + O(\ell/M + \ell/M_2)) \).

Now we must find an upper bound for the number of 4-tuples in \( S' \) which give an illegal reverse \( \ell \)-switching in \( Q \), and subtract this number from \( |S'| \). By Lemma 2.2 it suffices to find upper bounds for the number of elements of \( S' \) which satisfy (at least) one of conditions (I'), (II') or (III'). If Condition (I') holds then \( \tilde{W}_j \) contains a loop for some \( j \in \{1, 2\} \), which is true for \( O(\ell k_{\text{max}} M) \) 4-tuples in \( S' \). (Recall that \( \tilde{U} \) has no loop, by definition of \( S' \).) Condition (II') holds if \( \hat{e} \cap \hat{f}_j \) is nonempty for some \( j \in \{1, 2\} \). This occurs for at most \( O(k_{\text{max}} M_2) \) 4-tuples in \( S' \). Next, suppose that Condition (III') holds. Then there exists an edge \( e' \in G(Q') \setminus \{\hat{e}, \hat{f}_1, \hat{f}_2\} \) which intersects both \( \hat{e} \) and \( \hat{f}_j \) for some \( j \in \{1, 2\} \). The number of 4-tuples in \( S' \) which satisfy this condition is \( O(k_{\text{max}}^2 M_2) \).

Putting these contributions together, the number of 4-tuples in \( S' \) which give a legal reverse \( \ell \)-switchings from \( Q' \) is

\[
(r - 1)MM_2 \left( 1 + O \left( \frac{k_{\text{max}}^2 M}{M} + \frac{\ell k_{\text{max}}}{M_2} \right) \right) = (r - 1)MM_2 \left( 1 + O \left( \frac{k_{\text{max}}^3 + \ell k_{\text{max}}}{M_2} \right) \right), \tag{2.5}
\]

since \( 1/M \leq k_{\text{max}}/M_2 \). Combining (2.4) and (2.5) completes the proof. \( \square \)

The following summation lemma from [5] will be needed, and for completeness we state it here. (The statement has been adapted slightly from that given in [5], without affecting the proof given there.)

**Lemma 2.4** ([5, Corollary 4.5]). Let \( N \geq 2 \) be an integer and, for \( 1 \leq i \leq N \), let real numbers \( A(i) \), \( C(i) \) be given such that \( A(i) \geq 0 \) and \( A(i) - (i - 1)C(i) \geq 0 \). Define \( A_1 = \min_{i=1}^N A(i) \), \( A_2 = \max_{i=1}^N A(i) \), \( C_1 = \min_{i=1}^N C(i) \) and \( C_2 = \max_{i=1}^N C(i) \). Suppose that there exists a real number \( \hat{c} \) with \( 0 < \hat{c} < \frac{1}{3} \) such that \( \max\{A_2/N, |C_1|, |C_2|\} \leq \hat{c} \). Define \( n_0, \ldots, n_N \) by \( n_0 = 1 \) and

\[
n_i = \frac{1}{i} (A(i) - (i - 1)C(i)) n_{i-1}
\]

for \( 1 \leq i \leq N \). Then

\[
\Sigma_1 \leq \sum_{i=0}^N n_i \leq \Sigma_2,
\]

12
where
\[
\Sigma_1 = \exp\left( A_1 - \frac{1}{2} A_1 C_2 \right) - (2\hat{c})^N, \\
\Sigma_2 = \exp\left( A_2 - \frac{1}{2} A_2 C_1 + \frac{1}{2} A_2 C_2^2 \right) + (2\hat{c})^N. 
\]

This summation lemma will now be applied.

**Lemma 2.5.** Under the conditions of Theorem 1.1 we have
\[
\sum_{\ell=0}^{N} |C_\ell| = |C_0| \exp\left( \frac{(r - 1)M_2}{2M} + O\left( \frac{k_{\text{max}}^3}{M} \right) \right).
\]

**Proof.** Let \( \ell' \) be as defined in Lemma 2.3. By (2.4), any \( Q \in C_\ell \) can be converted to some \( Q' \in C_{\ell-1} \) using an \( \ell \)-switching. Hence \( C_\ell = \emptyset \) for \( \ell' \leq \ell \leq N \). In particular, the lemma holds if \( C_0 = \emptyset \), so we assume that \( \ell' \geq 1 \).

By Lemma 2.3, there exists some uniformly bounded function \( \beta_\ell \) such that
\[
\frac{|C_\ell|}{|C_0|} = \frac{1}{\ell} \frac{|C_{\ell-1}|}{|C_0|} (A(\ell) - (\ell - 1)C(\ell))
\]
for \( \ell = 1, \ldots, N \), where
\[
A(\ell) = \frac{(r - 1)M_2 - \beta_\ell k_{\text{max}}^3}{2M}, \quad C(\ell) = \frac{\beta_\ell k_{\text{max}}}{2M}
\]
for \( 1 \leq \ell < \ell' \), and \( A(\ell) = C(\ell) = 0 \) for \( \ell' \leq \ell \leq N \).

Now we apply Lemma 2.4. It is clear that \( A(\ell) - (\ell - 1)C(\ell) \geq 0 \), from (2.6) if \( 1 \leq \ell < \ell' \), or by definition if \( \ell' \leq \ell \leq N \). If \( \beta_\ell \geq 0 \) then \( A(\ell) \geq A(\ell) - (\ell - 1)C(\ell) \geq 0 \), while if \( \beta_\ell < 0 \) then \( A(\ell) \) is nonnegative by definition. Next, define \( A_1, A_2, C_1, C_2 \) to be the minimum and maximum of \( A(\ell) \) and \( C(\ell) \) over \( 1 \leq \ell \leq N \), as in Lemma 2.4, and set \( \hat{c} = \frac{1}{16} \). Since \( A_2 = (r - 1)M_2/(2M) + o(1) \) and \( C_1, C_2 = o(1) \), we have that \( \max\{A_2/N, |C_1|, |C_2|\} \leq \hat{c} \) for \( M \) sufficiently large, by definition of \( N \). Lemma 2.4 applies and gives an upper bound
\[
\sum_{\ell=0}^{N} \frac{|C_\ell|}{|C_0|} \leq \exp\left( \frac{(r - 1)M_2}{2M} + O\left( \frac{k_{\text{max}}^3}{M} \right) \right) + O((e/8)^N).
\]
Now \((e/8)^N \leq (e/8)^{\log M} \leq M^{-1}\), which leads to
\[
\sum_{\ell=0}^{N} \frac{|C_\ell|}{|C_0|} \leq \exp\left( \frac{(r - 1)M_2}{2M} + O\left( \frac{k_{\text{max}}^3}{M} \right) \right). \tag{2.7}
\]
If \( \ell' = N + 1 \) then the lower bound given by Lemma 2.4 is the same as the upper bound (2.7), within the stated error term, establishing the result in this case.

Finally suppose that \( 1 \leq \ell' \leq N \). Then (2.5) shows that

\[
M_2 = O(k_{\text{max}}^3 + \ell' k_{\text{max}}) = o(M + M^{1/3} \log M) = o(M).
\]

If \( \ell' = 1 \) then \( M_2 = O(k_{\text{max}}^3) \) and hence \((r - 1)M_2/(2M) = O(k_{\text{max}}^3/M)\), so in this case the trivial lower bound of 1 matches the upper bound (2.7), within the stated error term. If \( 2 \leq \ell' \leq N \) then using (2.6) with \( \ell = 1 \), we obtain

\[
\sum_{\ell=0}^{N} \frac{|C_\ell|}{|C_0|} \geq 1 + \frac{|C_1|}{|C_0|} = 1 + A(1) = 1 + \frac{(r - 1)M_2}{2M} + O(k_{\text{max}}^3/M).
\]

Since here \( M_2 = o(M) \), this expression matches the upper bound (2.7), within the stated error term. This completes the proof.

Theorem 1.1 now follows immediately, by combining (1.2), (2.1) and Lemma 2.5.

References


