A threshold result for loose Hamiltonicity in random regular uniform hypergraphs

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Abstract

Let \( G(n,r,s) \) denote a uniformly random \( r \)-regular \( s \)-uniform hypergraph on \( n \) vertices, where \( s \) is a fixed constant and \( r = r(n) \) may grow with \( n \). An \( \ell \)-overlapping Hamilton cycle is a Hamilton cycle in which successive edges overlap in precisely \( \ell \) vertices, and 1-overlapping Hamilton cycles are called loose Hamilton cycles.

When \( r,s \geq 3 \) are fixed integers, we establish a threshold result for the property of containing a loose Hamilton cycle. This partially verifies a conjecture of Dudek, Frieze, Ruciński and Šileikis (2015). In this setting, we also find the asymptotic distribution and expected value of the number of loose Hamilton cycles in \( G(n,r,s) \).

Finally we prove that for \( \ell = 2, \ldots, s-1 \) and for \( r \) growing moderately as \( n \to \infty \), the probability that \( G(n,r,s) \) has a \( \ell \)-overlapping Hamilton cycle tends to zero.

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1 Introduction

A hypergraph $G = (V,E)$ consists of a finite set $V$ of vertices and a multiset $E$ of multisubsets of $V$, which we call edges. We say that $H$ is simple if $E$ is a set of sets: that is, there are no repeated edges and no edge contains a repeated vertex. Given a fixed integer $s \geq 2$, the hypergraph $G$ is said to be $s$-uniform if every edge contains precisely $s$ vertices, counting multiplicities. Uniform hypergraphs have been well-studied, as they generalise graphs (which are 2-uniform hypergraphs). Let $r \geq 1$ be an integer. A hypergraph is said to be $r$-regular if every vertex has degree $r$, counting multiplicities.

(For more background on hypergraphs, see [3].)

For integers $r, s \geq 2$, let $S(n,r,s)$ be the set of all simple, $r$-regular, $s$-uniform hypergraphs on the vertex set $\{1,2,\ldots,n\}$. To avoid trivialities, assume that $s$ divides $rn$ (as any hypergraph in $S(n,r,s)$ has $rn/s$ edges). We write $\mathcal{G}(n,r,s)$ to denote a random hypergraph chosen uniformly from $S(n,r,s)$. Here $s$ is fixed, though we sometimes allow $r = r(n)$ to grow with $n$.

There are many ways to define cycles in hypergraphs. Most generally, for $k \geq 2$, a $k$-cycle is a set of $k$ edges which can be labelled as $e_0, e_1, \ldots, e_{k-1}$ such that $e_j \cap e_{j+1}$ is nonempty for $j = 0, \ldots, k-1$ (indices taken modulo $k$). A 1-cycle is an edge which contains a repeated vertex. Hence a hypergraph is simple if and only if it contains no 1-cycle and no 2-cycle consisting of two identical edges. A $k$-cycle is called a loose $k$-cycle if

$$|e_i \cap e_j| = \begin{cases} 1 & \text{if } |i - j| = 1 \mod k, \\ 0 & \text{otherwise}. \end{cases}$$

A loose $k$-cycle $C$ contains precisely $k(s-1)$ vertices. Figure 1 shows a loose 6-cycle in a 3-uniform hypergraph (the other edges of the hypergraph are not shown).

![Figure 1: A loose 6-cycle in a 3-uniform hypergraph.](image)

A loose 1-cycle is a 1-cycle which contains exactly $s - 1$ distinct vertices; that is, it contains one vertex $v$ with multiplicity 2, while the remaining $s - 2$ vertices are distinct from $v$ and from each other.

Define $t = n/(s-1)$ and let $G$ be a hypergraph on $n$ vertices. Observe that a loose $t$-cycle of $G$ covers all $(s-1)t = n$ vertices of $G$. From now on, we refer to a loose
A necessary condition for the existence of a loose Hamilton cycle in a hypergraph $G \in \mathcal{S}(n, r, s)$ is that $s - 1$ divides $n$.

More generally, an $\ell$-overlapping Hamilton cycle is a set of $t_\ell = n/(s - \ell)$ edges which can be labelled $e_0, e_1, \ldots, e_{t_\ell - 1}$ such that for some ordering $v_0, \ldots, v_{n-1}$ of the vertices we have

$$e_i = \{v_{i(s-\ell)}, v_{i(s-\ell)+1}, \ldots, v_{i(s-\ell)+s-1}\} \quad \text{for} \quad i = 0, \ldots, t_\ell - 1. \quad (1.1)$$

Here the vertex labels are also interpreted cyclically, so that for example

$$e_{t_\ell - 1} = \{v_{n-s+\ell}, v_{n-s+\ell+1}, \ldots, v_{n-1}, v_0, v_1, \ldots, v_{\ell-1}\}.$$

A necessary condition for an $\ell$-overlapping Hamilton cycle to exist in an $s$-uniform hypergraph on $n$ vertices is that $s - \ell$ divides $n$. An $(s - 1)$-overlapping Hamilton cycle is also called a tight Hamilton cycle, and a 1-overlapping Hamilton cycle is just a loose Hamilton cycle.

In this paper, asymptotic results for $\ell$-overlapping Hamilton cycles hold as $n \to \infty$ restricted to the set

$$\mathcal{I}^{(\ell)}_{(r,s)} = \{n \in \mathbb{Z}^+ : s \mid rn \text{ and } s - \ell \mid n\}.$$ 

If the probability of an event tends to 1 as $n \to \infty$ along this set, then we say that the event holds asymptotically almost surely (a.a.s.). Since our main focus is on loose Hamilton cycles, we write $\mathcal{I}_{(r,s)}$ instead of $\mathcal{I}^{(1)}_{(r,s)}$ when $\ell = 1$.

The case $s = 2$ (graphs) has been extensively studied. In order to prove that random $r$-regular graphs are a.a.s. Hamiltonian, for fixed $r \geq 3$, Robinson and Wormald [12, 13] used an analysis of variance technique now known as the small subgraph conditioning method. In [12], Robinson and Wormald proved that random cubic graphs are a.a.s. Hamiltonian, but their generalisation [13] to higher degrees used an inductive argument based on the a.a.s. presence of perfect matchings in random regular graphs of degree $r \geq 4$. The ideas of [12, 13] were further developed by Frieze et al. [8] and by Janson [11]. Frieze et al. [8] provided algorithmic results for the construction, generation and counting of Hamilton cycles in random regular graphs, while Janson [11] applied small subgraph conditioning to give the asymptotic distribution of the number of Hamilton cycles in random $r$-regular graphs. Janson stated this distribution in [11, Theorem 2], and noted that in particular, the expected number of Hamilton cycles in random $r$-regular graphs is asymptotically equal to

$$e \sqrt{\frac{\pi}{2r}} \left(\frac{(r-2)^{(r-2)/2} (r-1)}{r^{(r-2)/2}}\right)^n. \quad (1.2)$$

Janson also observed that [11, Theorem 2] directly implies that random $r$-regular graphs are a.a.s. Hamiltonian, when $r \geq 3$. 

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Our aim in this paper is to extend the results of Frieze et al. [8] and Janson [11] by using small subgraph conditioning to study loose Hamilton cycles in random \( s \)-uniform \( r \)-regular hypergraphs, for any \( r, s \geq 2 \). Our work is also motivated by two conjectures stated by Dudek et al [2], as discussed in Section 1.1. Where possible, we state our results so that they also cover the known results for graphs (\( s = 2 \)).

**Theorem 1.1.** Let \( s \geq 2 \) be a fixed integer. There exists a positive constant \( \rho(s) \) such that for any integer \( r \geq 2 \), as \( n \to \infty \) along \( \mathcal{I}(r,s) \),

\[
\Pr(G(n,r,s) \text{ contains a loose Hamilton cycle}) \longrightarrow \begin{cases} 
1 & \text{if } r > \rho(s), \\
0 & \text{if } r \leq \rho(s).
\end{cases}
\]

Specifically, \( \rho = \rho(s) \) is the unique real number in \( [2, \infty) \) such that

\[
(r - 1)(s - 1) \left( \frac{ps - \rho - s}{\rho s - \rho} \right)^{(s-1)(\rho s - \rho - s)/s} = 1.
\]

We note that \( \rho(2) \in (2, 3) \) and \( \rho(3) = 3 \), while if \( s \geq 4 \) then

\[
\rho^-(s) < \rho(s) < \rho^+(s) \tag{1.3}
\]

where

\[
\rho^-(s) = \frac{e^{s-1}}{s-1} - \frac{s - 2}{2} - \frac{(s^2 - s + 1)^2}{se^{s-1}},
\]

\[
\rho^+(s) = \frac{e^{s-1}}{s-1} - \frac{s - 2}{2}.
\]

In Table 1 we give the values of \( \rho(s) \) for \( s = 2, \ldots, 10 \), and for \( s = 4, \ldots, 10 \) we compare \( \rho(s) \) with the lower and upper bounds \( \rho^-(s), \rho^+(s) \) given in (1.3). All values are rounded to three decimal places, except \( \rho(3) = 3 \), since it is an integer. (Three decimal places are required to see that \( \rho(5) < 12 \).) We see that \( \rho(s) \) is closely approximated by the upper bound \( \rho^+(s) \), except at very small values of \( s \).

<table>
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<th>( s )</th>
<th>2</th>
<th>3</th>
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<th>5</th>
<th>6</th>
<th>7</th>
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<th>9</th>
<th>10</th>
</tr>
</thead>
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<td>( \rho^-(s) )</td>
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<td>–</td>
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<td>63.974</td>
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<tr>
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<td>11.998</td>
<td>27.580</td>
<td>64.675</td>
<td>153.625</td>
<td>369.100</td>
<td>896.332</td>
<td></td>
</tr>
<tr>
<td>( \rho^+(s) )</td>
<td>–</td>
<td>–</td>
<td>5.695</td>
<td>12.150</td>
<td>27.683</td>
<td>64.738</td>
<td>153.662</td>
<td>369.120</td>
<td>896.342</td>
</tr>
</tbody>
</table>

**Table 1:** Values of \( \rho(s) \) for small \( s \), together with our bounds (for \( s \geq 4 \)).

Let \( Y_G \) be the number of loose Hamilton cycles in \( G(n,r,s) \). When \( s \geq 3 \) and \( r > \rho(s) \), our calculations provide the asymptotic distribution of \( Y_G \), stated later as Theorem 6.2. Our calculations also lead to an asymptotic expression for the expected value of \( Y_G \), for any fixed \( r, s \geq 2 \) with \((r, s) \neq (2, 2)\).
Theorem 1.2. Suppose that $r, s \geq 2$ are fixed integers with $(r, s) \neq (2, 2)$. Then as $n \to \infty$ along $I_{(r, s)}$, 

$$
EY_G \sim \exp \left( \frac{(s-1)(rs-s-2)}{2(rs-r-s)} \right) \sqrt{\frac{\pi}{2n}}(s-1) \times \left( (r-1)(s-1) \left( \frac{rs-r-s}{rs-r} \right)^{(s-1)(rs-r-s)/s} \right)^{n/(s-1)}.
$$

Observe that this expectation tends to zero whenever $r \leq \rho(s)$. In particular, this implies the negative side of our threshold result, Theorem 1.1.

When $s = 2$ and $r \geq 3$, Theorem 1.2 matches the expression proved by Janson [11, Theorem 2], quoted above in (1.2). When $r = s = 2$, a 2-regular graph is Hamiltonian if and only if it is connected, and a Hamiltonian 2-regular graph contains exactly one Hamilton cycle. Hence in this case, $EY_G$ equals the probability that a random 2-regular graph is Hamiltonian, and Wormald [14, Equation (11)] noted that this probability is asymptotically equal to

$$
\frac{e^{3/4}}{2} \sqrt{\frac{\pi}{n}}.
$$

Our final result concerns $\ell$-overlapping Hamilton cycles and allows the degree $r$ to grow moderately with $n$.

**Theorem 1.3.** Let $s \geq 3$ be a fixed integer and let

$$
\kappa = \kappa(s) = \begin{cases} 
1 & \text{if } s \geq 4, \\
1/2 & \text{if } s = 3.
\end{cases}
$$

Suppose that $r = r(n)$ with $3 \leq r = o(n^\kappa)$. Then as $n \to \infty$ along $I_{(r, s)}^{(\ell)}$, a.a.s. $G(n, r, s)$ has no $\ell$-overlapping Hamilton cycle for $\ell = 2, \ldots, s-1$.

To prove these results, as is usual in this area, we will work in a related probability model known as the configuration model. After discussing some related results and extensions in Section 1.1, we review the configuration model for hypergraphs in Section 2 and prove Theorem 1.3 in Section 2.1. To prove our remaining results we will apply the small subgraph conditioning method, which is discussed in Section 2.2. The structure of the rest of the paper will be described in Section 2.3.

### 1.1 Extensions and related results

The small subgraph conditioning method has been applied to prove many a.a.s. structural theorems (contiguity results) for regular graphs. See Wormald [14] or Janson [11] for more detail. For uniform regular hypergraphs, we only know of one application of
the method: Cooper et al. [2] used small subgraph conditioning to investigate perfect matchings in random regular uniform hypergraphs. They proved a threshold result for existence of a perfect matching in a random $r$-regular $s$-uniform hypergraph, where $r, s \geq 2$ are fixed integers. Specifically, in [2, Theorem 1] they proved that as $n \to \infty$, this probability tends to 0 if $s > \sigma_r$ and tends to 1 if $s < \sigma_r$, where

$$\sigma_r = \frac{\ln r}{(r - 1) \ln \left( \frac{r}{r-1} \right)} + 1.$$  

Defining $r_0(s) = \min\{r : s < \sigma_r\}$, Cooper et al. [2] remark that $r_0(s)$ is approximately $e^{s-1}$. It is interesting to observe that, in contrast to graphs ($s = 2$), the threshold for the existence of perfect matchings in $\mathcal{G}(n, r, s)$ is higher than the threshold for the existence of loose Hamilton cycles when $s \geq 3$; that is, $r_0(s) > \rho(s)$.

Recently, Dudek et al. [6] established a relation between $\mathcal{G}(n, r, s)$ and the uniform probability model on the set of $s$-uniform hypergraphs on $n$ vertices with $m$ edges (when $s \geq 3$). Using known results about the existence of loose Hamilton cycles in the latter model, they showed that a.a.s. $\mathcal{G}(n, r, s)$ contains a loose Hamilton cycle when $r \gg \ln n$ (or $r = \Omega(\ln n)$, if $s = 3$) and $r = o(n^{1/2})$.

Dudek et al. made the following conjecture [6, Conjecture 1], rewritten here in our notation:

For every $s \geq 3$ there exists a constant $\rho = \rho(s)$ such that for any $r \geq \rho$, $\mathcal{G}(n, r, s)$ contains a loose Hamilton cycle a.a.s.

We have partially verified this conjecture with Theorem 1.1, which gives a threshold result for constant values of $r$. This leaves a gap for degrees $r = r(n) = O(\ln n)$. Intuitively, it seems that increasing the degree should make the existence of a loose Hamilton cycle more likely. However, as the small subgraph conditioning method does not apply directly when $r$ is growing, it appears that other ideas are required, such as a switching argument.

There has been much work on $\ell$-overlapping Hamilton cycles in the binomial model $\mathcal{G}_{n,p}^{(s)}$ of $s$-uniform hypergraphs, where each $s$-set is an edge with probability $p$, independently. In particular, loose ($\ell = 1$) and tight ($\ell = s - 1$) Hamilton cycles are well studied, see for example [1, 4, 7] and references therein. Dudek and Frieze [4, Theorem 3(i)] proved that for all fixed integers $s > \ell \geq 2$ and fixed $\epsilon > 0$, if $p \leq (1 - \epsilon)e^{s-\ell}/n^{s-\ell}$ then a.a.s. $\mathcal{G}_{n,p}^{(s)}$ has no $\ell$-overlapping Hamilton cycle. This motivated the second conjecture of Dudek et al. [6, Conjecture 2], written here in our notation:

For every $s > \ell \geq 2$, if $r \gg n^{\ell-1}$ then a.a.s. $\mathcal{G}(n, r, s)$ contains an $\ell$-overlapping Hamilton cycle.

Theorem 1.3 can be seen as a contribution towards the proof of the corresponding negative result needed to establish a degree threshold for the existence of an $\ell$-overlapping Hamilton cycle. We believe that complex-analytic methods such as those presented in [10] may allow progress towards the proof of [6, Conjecture 2], and will investigate this in future work.
2 Main ideas

We work in a natural generalisation of the configuration model to hypergraphs. This is the same model used by Cooper et al. [2]. We use the notation $[n] = \{1, 2, \ldots, n\}$.

Let $B_1, B_2, \ldots, B_n$ be disjoint sets of size $r$, which we call cells, and define $\mathcal{B} = \bigcup_{i=0}^{n} B_i$. Elements of $\mathcal{B}$ are called points. We assume that there is a fixed ordering on the $rn$ points of $\mathcal{B}$ (so that different points in the same cell are distinguishable).

Assume that $s$ divides $rn$. Let $\Omega(n, r, s)$ be the set of all unordered partitions $F = \{U_1, \ldots, U_{rn/s}\}$ of $\mathcal{B}$ into $rn/s$ parts, where each part has exactly $s$ points. Each partition $F \in \Omega(n, r, s)$ defines a hypergraph $G(F)$ on the vertex set $[n]$ in a natural way: vertex $i$ corresponds to the cell $B_i$, and each part $U \in F$ gives rise to an edge $e_U$ such that the multiplicity of vertex $i$ in $e_U$ equals $|U \cap B_i|$, for $i = 1, \ldots, n$. Then $G(F)$ is an $s$-uniform $r$-regular hypergraph. The partition $F \in \Omega(n, r, s)$ is called simple if $G(F)$ is simple. More generally, we will often describe $F \in \Omega(n, r, s)$ as having a particular hypergraph property if $G(F)$ has that property.

For any positive integer $\ell$ which is divisible by $s$, define

$$p(\ell) = \frac{\ell!}{(\ell/s)! (s!)^{\ell/s}}.$$  \hfill (2.1)

Then

$$|\Omega(n, r, s)| = p(rn) = \frac{(rn)!}{(rn/s)! (s!)^{rn/s}}.$$  \hfill (2.2)

Every hypergraph in $\mathcal{S}(n, r, s)$ corresponds to precisely $(r!)^n$ partitions $F \in \Omega(n, r, s)$. (This is only true for simple hypergraphs.) Therefore

$$|\mathcal{S}(n, r, s)| = \frac{|\Omega(n, r, s)| \Pr(\mathcal{F}(n, r, s) \text{ is simple})}{(r!)^n},$$  \hfill (2.3)

where $\mathcal{F}(n, r, s)$ denotes a random partition chosen uniformly from $\Omega(n, r, s)$. Observe also that $\mathcal{G}(n, r, s)$ has the same distribution as $G(\mathcal{F}(n, r, s))$, conditioned on the event that $\mathcal{F}(n, r, s)$ is simple.

It was shown in [2] that when $r, s \geq 2$ are fixed,

$$\lim_{n \to \infty} \Pr(\mathcal{F}(n, r, s) \text{ is simple}) = e^{-(r-1)(s-1)/2}.$$  \hfill (2.4)

Now let $\mathcal{A}$ be any event with probability $o(1)$ in $\mathcal{F}(n, r, s)$. Then

$$\Pr(\mathcal{F}(n, r, s) \in \mathcal{A} | \mathcal{F}(n, r, s) \text{ is simple}) \leq \frac{\Pr(\mathcal{F}(n, r, s) \in \mathcal{A})}{\Pr(\mathcal{F}(n, r, s) \text{ is simple})} = o(1),$$  \hfill (2.5)

which implies that the corresponding event in $\mathcal{G}(n, r, s)$ also has probability $o(1)$. 

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2.1 Expected value in configuration model

For $\ell = 1, \ldots, s - 1$, let $Y^{(\ell)}$ be the number of $\ell$-overlapping Hamilton cycles in $\mathcal{F}(n, r, s)$. We now calculate the expected value of $Y^{(\ell)}$.

Lemma 2.1. Let $\ell \in \{1, \ldots, s - 1\}$, where $s \geq 3$ is fixed, and let $r = r(n)$ be a function of $n$ which satisfies $r \geq 3$. Write $\ell = q(s - \ell) + c$ where $q, c$ are nonnegative integers and $c \in \{0, 1, \ldots, s - \ell - 1\}$. Then as $n \to \infty$ along $\mathcal{I}_{(r,s)}^{(\ell)}$,

$$
\mathbf{E}Y^{(\ell)} \sim \sqrt{\frac{\pi}{2n}} (s - \ell) \left( \frac{c}{n} \right)^{(\ell-1)n/(s-\ell)} \left( \frac{(r)_q}{c!} \right)^{s-\ell} \left( \frac{(r - q - 1)_c}{c!} \right)^{s-\ell} \left( \frac{(s - 1)_{c+\ell-1}}{c!} \right)^{s-\ell} \left( \frac{(rs - r\ell - s)_{s-1}(rs - r\ell - s)/s}{c!} \right)^{n/(s-\ell)}.
$$

Proof. Recall the notation $t_\ell = n/(s - \ell)$. There are $n!$ ways to fix an ordering $v_0, \ldots, v_{n-1}$ of the vertices. This gives rise to an $\ell$-overlapping Hamilton cycle $H$ with edges $e_0, \ldots, e_{t_\ell-1}$ defined by (1.1). For $j = 0, \ldots, t_\ell - 1$ define the set $S_j = e_j \setminus e_{j-1}$, with index arithmetic performed cyclically (so $S_0 = e_0 \setminus e_{t_\ell-1}$). The sets $S_0, \ldots, S_{t_\ell-1}$ partition $[n]$.

First, observe that $s - \ell - c$ vertices in $S_i$ have degree $q + 1$ in the Hamilton cycle $H$, and the remaining $c$ vertices of $S_i$ have degree $q + 2$ in $H$, for all $i \in \{0, 1, \ldots, t_\ell - 1\}$. It follows that the chosen Hamilton cycle $H$ (as a hypergraph) corresponds to precisely $2t_\ell (c! (s - \ell - c)!)^{t_\ell}$ orderings of the vertices.

The number of ways to embed the edges of $H$ as parts in a partition $F$ is

$$
\left( ((r)_q+1)^{s-\ell-c} ((r)_q+2)^c \right)^{t_\ell} = \left( ((r)_q+1)^{s-\ell} (r - q - 1)^c \right)^{t_\ell},
$$

This completely specifies the $t_\ell$ parts of the partition which correspond to $H$. Finally, we must multiply by

$$
\frac{p(rn - st_\ell)}{p(rn)}
$$

for the probability that a randomly chosen partition contains these $t_\ell$ specified parts. We have shown that

$$
\mathbf{E}Y^{(\ell)} = \frac{n!}{2t_\ell (c! (s - \ell - c)!)^{t_\ell}} \left( ((r)_q+1)^{s-\ell} (r - q - 1)^c \right)^{t_\ell} \frac{p(rn - st_\ell)}{p(rn)}
$$

$$
= \frac{n!}{2t_\ell (c! (s - \ell - c)!)^{t_\ell}} \left( ((r)_q+1)^{s-\ell} (r - q - 1)^c \right)^{t_\ell} \frac{(rn - st_\ell)! (rn/s)! (s!)^{rn/s}}{(rn/s - t_\ell)! (s!)^{rn/s-t_\ell} (rn)!},
$$

and the result follows by applying Stirling’s formula. \qed
At this point we can also prove Theorem 1.3.

**Proof of Theorem 1.3.** Recall from the statement of Theorem 1.3 that \( \kappa = \kappa(s) \) equals 1 if \( s \geq 4 \), and equals \( \frac{1}{2} \) when \( s = 3 \). Fix \( \ell = 2, \ldots, s-1 \), where \( s \geq 3 \) is constant and \( r = r(n) \) may grow with \( n \), such that \( 3 \leq r = o(n^\kappa) \). It follows from Lemma 2.1, by definition of \( q, c \), that

\[
\mathbb{E}Y^{(\ell)} \leq \left( \frac{c'r}{n^{\ell-1}} \right)^{n/(s-\ell)}
\]

for some positive constant \( c' \) (independent of \( r \)). Dudek et al. [5, Theorem 1] proved that when \( r = o(n^\kappa) \),

\[
|S(n, r, s)| = \frac{(rn)!}{(rn/s)! (s!)^{rn/s} (r!)^n} \exp \left( -\frac{1}{2} (r-1)(s-1) + O( (r/n)^{1/2} + r^2/n ) \right).
\]

Combining this with (2.2) and (2.3), we conclude that when \( r = o(n^\kappa) \),

\[
\Pr(\mathcal{F}(n, r, s) \text{ is simple}) = \exp \left( -\frac{1}{2} (r-1)(s-1)(1 + o(1)) \right) = \Omega(\exp(-\hat{c}r))
\]

for some positive constant \( \hat{c} \) (independent of \( r \)). By (2.5) and Markov’s Lemma, it follows that the probability that \( G \in \mathcal{G}(n, r, s) \) contains an \( \ell \)-overlapping Hamilton cycle is bounded above by

\[
\frac{\mathbb{E}Y^{(\ell)}}{\Pr(\mathcal{F}(n, r, s) \text{ is simple})} \leq O \left( \exp(\hat{c}r) \right) \left( \frac{c'r}{n^{\ell-1}} \right)^{n/(s-\ell)}

= O(1) \exp \left( \hat{c}r - \frac{n}{s-\ell} \ln \left( \frac{n^{\ell-1}}{c'r} \right) \right)
\]

which is \( o(1) \) for any \( r = o(n) \). This completes the proof. \( \square \)

We are particularly interested in loose Hamilton cycles (\( \ell = 1 \)) for fixed \( r \). Recall that \( Y = Y^{(1)} \) is the number of loose Hamilton cycles in \( \mathcal{F}(n, r, s) \). The next result follows from substituting \( \ell = 1 \) into Lemma 2.1.

**Corollary 2.2.** Let \( r, s \geq 3 \) be fixed integers and let \( t = n/(s-1) \). The expected value of \( Y \) satisfies

\[
\mathbb{E}Y = \frac{n!}{2t ((s-2)!)^t} \cdot r^n (r-1)^t \cdot \frac{p(rn-st)}{p(rn)} \tag{2.6}
\]

\[
\sim \sqrt{\frac{\pi}{2n}} (s-1) \left( (r-1)(s-1) \left( \frac{rs-r-s}{rs-r} \right)^{(s-1)(rs-r-s)/s} \right)^{n/(s-1)}
\]

as \( n \to \infty \) along \( \mathcal{I}_{(r,s)} \).

We can now prove that Theorem 1.2 holds in the special case that \( r = 2 \) and \( s = 3 \).
Lemma 2.3. The conclusion of Theorem 1.2 holds when \( r = 2 \) and \( s = 3 \).

Proof. Suppose that \( F \in \Omega(n, 2, 3) \) contains a set of parts \( F_H \) such that \( G(F_H) \) is a Hamilton cycle. Then \( n/2 \) vertices of \( G(F) \) have degree 2 in the subgraph \( G(F_H) \), and so are contained in no other edge of \( G(F) \). Hence, \( G(F \setminus F_H) \) forms a perfect matching of the remaining \( n/2 \) vertices. In particular, there is no cycle in \( G(F \setminus F_H) \). Therefore, for \( F \in \Omega(n, 2, 3) \), if \( G(F) \) contains a Hamilton cycle then \( G(F \setminus F_H) \) forms a perfect matching of the remaining \( n/2 \) vertices. In particular, there is no cycle in \( G(F \setminus F_H) \). Therefore, for \( F \in \Omega(n, 2, 3) \), if \( G(F) \) contains a Hamilton cycle then \( G(F) \) is simple. Then

\[
\mathbb{E}Y = \sum_H \Pr(H \subseteq G(F(n, 2, 3)))
\]

\[
= \sum_H \frac{|\{F \in \Omega(n, 2, 3) : H \subseteq G(F)\}|}{|\Omega(n, 2, 3)|}
\]

\[
= \Pr(F(n, 2, 3) \text{ is simple}) \sum_H \frac{|\{F \in \Omega(n, 2, 3) : H \subseteq G(F) \text{ and } F \text{ is simple}\}|}{|\Omega(n, 2, 3)|}
\]

\[
= \Pr(F(n, 2, 3) \text{ is simple}) \mathbb{E}Y_G,
\]

where the sum is over all loose 3-uniform Hamilton cycles on \([n]\). Combining this with (2.4) and Corollary 2.2, it follows that the asymptotic expression for \( \mathbb{E}Y_G \) holds when \( r = 2 \) and \( s = 3 \).

In Lemma 6.1 we characterise pairs \((r, s)\) for which \( \mathbb{E}Y \) tends to infinity, leading to the definition of the threshold function \( \rho(s) \). Combining this result with (2.5), we obtain the negative part of the threshold result Theorem 1.1, as explained in Section 6. In order to complete the proof of Theorem 1.1, and prove Theorem 1.2, we require more information about the asymptotic distribution of the number of loose Hamilton cycles in \( F(n, r, s) \). This information is obtained using the small subgraph conditioning method.

2.2 Small subgraph conditioning for hypergraphs

The following statement of the small subgraph conditioning method is adapted from [11, Theorem 1]. A similar theorem is given in [14, Theorem 4.1].

Theorem 2.4 ([11]). Let \( \lambda_k > 0 \) and \( \delta_k \geq -1 \), \( k = 1, 2, \ldots \), be constants and suppose that for each \( n \) there are random variables \( X_{k,n} \), \( k = 1, 2, \ldots \), and \( Y_n \) (defined on the same probability space) such that \( X_{k,n} \) is non-negative integer valued and \( \mathbb{E}Y_n \neq 0 \) and furthermore the following conditions are satisfied:

(A1) \( X_{k,n} \overset{d}{\to} Z_k \) as \( n \to \infty \), jointly for all \( k \), where \( Z_k \sim \text{Po}(\lambda_k) \) are independent Poisson random variables;

(A2) For any finite sequence \( x_1, x_2, \ldots, x_m \) of non-negative integers,

\[
\frac{\mathbb{E}(Y_n | X_{1,n} = x_1, X_{2,n} = x_2, \ldots, X_{m,n} = x_m)}{\mathbb{E}Y_n} \to \prod_{k=1}^m (1 + \delta_k)^{x_k} e^{-\lambda_k \delta_k} \quad \text{as } n \to \infty;
\]
\[ \sum_{k \geq 1} \lambda_k \delta_k^2 < \infty; \]

(A4) \[ \frac{\mathbb{E}(Y^2_n)}{\mathbb{E}(Y^2_n)} \to \exp \left( \sum_k \lambda_k \delta_k^2 \right) \text{ as } n \to \infty. \]

Then
\[ \frac{Y_n}{\mathbb{E}Y_n} \xrightarrow{d} W = \prod_{k=1}^{\infty} (1 + \delta_k)^{Z_k} e^{-\lambda_k \delta_k} \text{ as } n \to \infty; \tag{2.7} \]
moreover, this and the convergence (A1) hold jointly. The infinite product defining \( W \) converges a.s. and in \( L^2 \), with
\[ \mathbb{E}W = 1 \text{ and } \mathbb{E}W^2 = \exp \left( \sum_{k \geq 1} \lambda_k \delta_k^2 \right) = \lim_{n \to \infty} \frac{\mathbb{E}(Y^2_n)}{\mathbb{E}(Y^2_n)}. \]

Furthermore, if \( \delta_k > -1 \) for all \( k \) then a.a.s. \( Y_n > 0 \).

Janson remarks in [11] that in the asymptotics, the index set \( \mathbb{Z}^+ \) may be replaced by any other countably-infinite set. The same is true for the other results stated in this section.

We emphasise that for the remainder of the paper, \( r \geq 3 \) is a fixed integer. Recall that \( t = n/(s - 1) \) is the number of edges in a loose Hamilton cycle. We will apply Theorem 2.4 to the random variables defined as follows. In order to distinguish our specific random variables from the general random variables used in Theorem 2.4, we do not include the subscript \( n \) in our notation.

- Let \( Y \) be the number of subsets \( F_H \) of \( \mathcal{F}(n, r, s) \) consisting of \( t \) parts such that \( G(F_H) \) is a loose Hamilton cycle.
- For \( k \geq 2 \) let \( X_k \) be the number of subsets \( F_C \) of \( \mathcal{F}(n, r, s) \) consisting of \( k \) parts such that \( G(F_C) \) is a loose \( k \)-cycle.
- Let \( X_1 \) be the number of parts \( U \) in \( \mathcal{F}(n, r, s) \) such that \( U \) gives rise to an edge which contains a repeated vertex. That is, \( X_1 \) is the number of parts \( U \) in \( \mathcal{F}(n, r, s) \) such that \( |U \cap B_j| > 1 \) for some \( j \in [n] \).

Note that \( X_1 \) counts parts which correspond to 1-cycles, not just loose 1-cycles. We define \( X_1 \) in this way so that \( X_1 = 0 \) if and only if no edge of \( G(F) \) contains a repeated vertex. (Our definition of \( X_1 \) agrees with that used in [2].)

Cooper et al. proved in [2, Section 5] that \( X_k \to Z_k \) as \( n \to \infty \), jointly for \( k \geq 1 \), where \( Z_k \sim \text{Po}(\lambda_k) \) are asymptotically independent Poisson random variables with mean
\[ \lambda_k = \frac{(r-1)(s-1)^k}{2k}. \tag{2.8} \]
This verifies that (A1) of Theorem 2.4 holds. In fact, Cooper et al. [2] worked with the random variable $X'_k$ for $k \geq 1$ which counts the number of $k$-cycles (not necessarily loose). Here a sequence of $k$ edges forms a $k$-cycle if successive pairs of edges overlap in at least one vertex (including the first and last edge of the sequence). Note that $X'_1 = X_1$. Calculations from [2, Section 5] show that $X'_k \sim X_k$ jointly for $k \geq 1$, since a.a.s. the contribution to $X_k$ from non-loose $k$-cycles forms only a negligible fraction of $X_k$. Here we write $A_n \sim B_n$ to mean that two sequences of random variables $(A_n)$ and $(B_n)$ have the same asymptotic distribution, recalling that both $X_k$ and $X'_k$ depend on $n$. Hence Theorem 2.4 (A1) holds with $\lambda_k$ as in (2.8).

In order to establish (A2) of Theorem 2.4, the following result (for general random variables) is convenient.

**Lemma 2.5** ([11, Lemma 1]). Let $\lambda'_k \geq 0$, $k = 1, 2, \ldots$, be constants. Suppose that (A1) holds, that $Y_n \geq 0$ and that

$$(A2')$$ for every finite sequence $x_1, x_2, \ldots, x_m$ of non-negative integers

$$\frac{\mathbb{E} (Y_n(X_{1,n})x_1(X_{2,n})x_2 \cdots (X_{m,n})x_m))}{\mathbb{E} Y_n} \to \prod_{k=1}^{m} (\lambda'_k)^{x_k} \text{ as } n \to \infty.$$

Then (A2) holds with $\lambda_k(1 + \delta_k) = \lambda'_k$ for all $k \geq 1$.

It is routine to extend the arguments of Section 3.2 and Section 4 to show that

$$\frac{\mathbb{E} (Y(X_{1})x_1(X_{2})x_2 \cdots (X_{m})x_m))}{\mathbb{E} Y} = (1 + o(1)) \prod_{k=1}^{m} \left( \frac{\mathbb{E}(YX_k)}{\mathbb{E} Y} \right)^{x_k},$$  \hspace{1cm} (2.9)

similarly to the case of Hamilton cycles in cubic graphs [12, equation (28)]. Roughly, (2.9) holds because fixing a constant-length cycle has asymptotically negligible effect on the number of ways to choose subsequent cycles, and noting that overlapping cycles also give negligible relative contribution. We omit these technical details, as is usual in the literature.

As Cooper et al. remark in [2, Section 2], the probability that two parts of $F \in \mathcal{F}(n, r, s)$ give rise to a repeated edge is $o(1)$; see also (4.6). Therefore the probability that $G(F)$ is simple is asymptotically equal to the probability that $Z_1 = 0$, which is $e^{-\lambda_1}$; see (2.4). Furthermore, the random variable $Y_{G}$ has the same asymptotic distribution as that of $\tilde{Y}$, where $\tilde{Y}$ is the the random variable obtained from $Y$ by conditioning on the event $X_1 = 0$. In particular, $\mathbb{E} Y_{G} \sim \mathbb{E} \tilde{Y} = \mathbb{E}(Y \mid X_1 = 0)$.

We now explain how the asymptotic distribution of $Y_{G}$ (Theorem 6.2) can be obtained from the asymptotic distribution of $Y$, once the latter has been established using Theorem 2.4.
Lemma 2.6. Suppose that $Y$ and $X_k$ satisfy conditions (A1)–(A4) of Theorem 2.4. Then

$$
\frac{Y_G}{\mathbb{E}Y_G} \overset{d}{\rightarrow} \prod_{k=2}^{\infty} (1 + \delta_k) Z_k e^{-\lambda_k \delta_k} \quad \text{as} \quad n \to \infty.
$$

Moreover, if $\delta_k > -1$ for all $k \geq 1$ then a.a.s. $Y_G > 0$.

Proof. Let $\hat{Y}$ be as defined above, and let $\hat{W}$ be the random variable obtained from $W$ by conditioning on the event that $Z_1 = 0$. By Theorem 2.4 applied to $Y$, the convergence of (2.7) and the convergence of (A1) holds jointly. Therefore, after conditioning on the event $X_1 = 0$, which has non-vanishing probability, we have

$$
\frac{\hat{Y}}{\mathbb{E} \hat{Y}} \overset{d}{\rightarrow} \hat{W} \quad \text{and} \quad \frac{\mathbb{E} \hat{Y}}{\mathbb{E} \hat{Y}} \to \mathbb{E} \hat{W}.
$$

This implies that

$$
\frac{Y_G}{\mathbb{E}Y_G} \overset{d}{\sim} \frac{\hat{Y}}{\mathbb{E} \hat{Y}} \overset{d}{\rightarrow} \left( \lim_{n \to \infty} \frac{\mathbb{E} Y}{\mathbb{E} \hat{Y}} \right) \hat{W} = \frac{\hat{W}}{\mathbb{E} \hat{W}}.
$$

Finally,

$$
\hat{W} = e^{-\lambda_1 \delta_1} \prod_{k=2}^{\infty} (1 + \delta_k) Z_k e^{-\lambda_k \delta_k}.
$$

Theorem 2.4 states that $\mathbb{E} W = 1$, which implies that $\mathbb{E} \hat{W} = e^{-\lambda_1 \delta_1}$. The first statement follows.

Next, suppose that $\delta_k > -1$ for all $k \geq 1$. Then a.a.s. $Y > 0$, by the final statement of Theorem 2.4 applied to $Y$. The probability that $F$ is simple is bounded below by a constant when $n \to \infty$, as observed by Cooper et al. [2]; see (2.4). Hence we can apply (2.5) to obtain

$$
\Pr(Y_G = 0) = O(\Pr(Y = 0)) = o(1),
$$

completing the proof.

2.3 Structure of the rest of the paper

We assume that $r, s \geq 3$, since the results for $s = 2$ were proved by Frieze et al. [8] and Janson [11]. It remains to investigate the second moment of $Y$ and the interaction of $Y$ with short cycles.

Section 3 contains some terminology and preliminary results, and describes a common framework which we will use for the calculations in the following two sections.

In Section 4 we calculate $\mathbb{E}(YX_k)/\mathbb{E}Y$, using a generating function to assist in our calculations. For each $k \geq 1$ this determines the value of $\delta_k$ such that $\mathbb{E}(YX_k)/\mathbb{E}Y$ tends to $\lambda_k(1 + \delta_k)$, where $\lambda_k$ is defined in (2.8). Standard arguments imply that Theorem 2.4 (A2) also holds, which allows us to prove Theorem 1.2.
The remainder of the paper is devoted to completing the small subgraph conditioning argument to prove Theorem 1.1. In Section 4.1 we calculate \( \sum_{k=1}^{\infty} \lambda_k \delta_k^2 \), proving that assumption (A3) of Theorem 2.4 holds. Section 5 contains the analysis of the second moment \( \mathbb{E}(Y^2) \). Here we use Laplace summation to find an asymptotic expression for the second moment, proving that (A4) of Theorem 2.4 holds. This involves proving that a certain 4-variable real function has a unique maximum in a certain bounded convex domain. (This optimisation is performed in Section A.) Finally, the proof of Theorem 1.1 is completed in Section 6.

3 Terminology and common framework

For the small subgraph conditioning method, we need to calculate the second moment of \( Y \) and establish condition (A2) of Theorem 2.4. We now describe a common framework which we will use for these calculations, which will be completed in Sections 4 and 5.

Suppose that \( F_C, F_H \) are both subpartitions of some partition in \( \Omega(n, r, s) \), such that \( G(F_H) \) is a loose Hamilton cycle and \( G(F_C) \) is a loose \( k \)-cycle. In particular, \( |F_H| = t = n/(s-1) \) and \( |F_C| = k \). Write \( H \) for \( G(F_H) \) to \( C \) for \( G(F_C) \). We will be particularly interested in two extreme cases, namely, when \( k \) is constant or when \( k = t \). (In the latter case, \( C \) is also a loose Hamilton cycle.) In order to describe the common framework we will use for our calculations in these cases, we need some terminology. We will use Figure 2 as a running example: it shows a 12-cycle \( C \) in a 5-uniform hypergraph, and some edges of a Hamilton cycle \( H \).

Figure 2: A 12-cycle \( C \) with 5 edges in \( G(F_C \cap F_H) \), in three paths.

Suppose that there are \( k-a \) parts in \( F_C \cap F_H \), and that \( G(F_C \cap F_H) \) forms \( b \) connected components in \( G(H) \), each of which is a path. Then there are \( a \) parts in \( F_C \setminus F_H \). In our running example from Figure 2, the edges shown in bold belong to \( G(F_C \cap F_H) \). There are 5 of them, in 3 paths, so \( a = 12 - 5 = 7 \) and \( b = 3 \). The 7 edges of \( G(F_C \setminus F_H) \)
are shown as thin rectangles. The dashed lines indicate partial edges which belong to \( G(F_H \setminus F_C) \).

Let \( v \) be a vertex in a loose cycle \( C \). If \( v \) has degree 2 in \( C \) then we will say that \( v \) is \( C\text{-external} \) (or just \( \text{external} \), if no confusion can arise). Otherwise, \( v \) has degree 1 in \( C \) and we will say that it is \( C\text{-internal} \) (or just \( \text{internal} \)). A loose Hamilton cycle \( H \) has \( t \) external vertices and \((s - 2)t \) internal vertices. In Figure 2, vertices \( x_1 \) and \( x_2 \) (shown as large black circles) are \( C\text{-external} \) and \( H\text{-external} \). Vertices \( z_1, z_2, z_3 \) and \( z_4 \) (shown as small black circles) are \( C\text{-external} \) and \( H\text{-internal} \). Finally, vertices \( y_1, y_2, y_3 \) and \( y_4 \) (shown as large white circles) are \( H\text{-external} \) and \( C\text{-internal} \). It will be important to know whether a given vertex is external or internal in \( C \) and/or \( H \).

The edges of \( G(F_C \cap F_H) \) which start or end a component of \( G(F_C \cap F_H) \) will play a special role: we call these \( \text{terminal edges} \). If a component of \( G(F_C \cap F_H) \) has length at least two then it has two terminal edges, and each terminal edge contains precisely one \( C\text{-external} \) vertex which is incident with an edge of \( G(F_C \setminus F_H) \). Such a vertex is called a \( \text{connection vertex} \). In Figure 2 there is one component of \( G(F_C \cap F_H) \) which has more than one edge (and hence has two distinct terminal edges). The connection vertices for this component are \( x_1 \) and \( x_2 \).

On the other hand, if a component of \( G(F_C \cap F_H) \) has length 1 then it has only one terminal edge, containing two connection vertices. In Figure 2 there are two such components: one has connection vertices \( z_2 \) and \( x_2 \), and the other has connection vertices \( z_3 \) and \( z_4 \). We refer to components of \( G(F_C \cap F_H) \) of length one as \( 1\text{-components} \).

As we will see later, the two connection vertices in components of \( G(F_C \cap F_H) \) of length at least two are essentially independent, as far as our counting argument is concerned, since they belong to \( \text{distinct} \) terminal edges. This is not true for the \( 1\text{-components} \) in \( G(F_C \cap F_H) \), so we need to take special care with these. (This is the main reason why the hypergraph calculations are substantially more difficult than the graph case.)

### 3.1 Templates

Starting with a set of parts \( F_H \) corresponding to an arbitrary Hamilton cycle \( H \), we need to consider all possible choices for a set \( F_C \) of parts corresponding to a \( k\)-cycle \( C \) and then, all possible completions of \( F_H \cup F_C \) to a partition in \( \Omega(n, r, s) \). We will do this with the help of templates, which we now define.

Suppose that \( F_C \) is given. If \( F_C \cap F_H \) is neither empty nor equal to \( F_H \), then we fix a start-vertex \( v \) which is a \( C\text{-external} \) vertex that belongs to precisely one edge \( e \) of \( G(F_C \cap F_H) \). This uniquely determines a direction around \( C \) in which \( e \) is the first edge of \( C \) (starting from \( v \)). Otherwise, let \( v \) be any \( C\text{-external} \) vertex and choose an arbitrary direction around \( C \). Then the template \( \theta \) for \( F_C \), with respect to the chosen start-vertex and direction, is the sequence \( \theta = (\theta_1, \ldots, \theta_k) \in \{0, 1\}^k \) such that the \( j \)'th coordinate of \( \theta \) is 1 if and only if the \( j \)'th part of \( F_C \) belongs to \( F_H \). Note that, by definition, either \( \theta_1 = 1 \) and \( \theta_k = 0 \), or \( \theta_1 = \theta_2 = \cdots = \theta_k \).
Let $b$ be the number of indices $j \in \{1, \ldots, k\}$ such that $\theta_j = 1$ and $\theta_{j+1} = 0$. If $b \geq 1$ then the template determines a sequence $\ell = (\ell_1, \ldots, \ell_b)$ of lengths of the components of $G(\mathcal{F}_C \cap \mathcal{F}_H)$, in order around $C$ (from the given start-vertex, in the given direction). Here the start-vertex of $C$ is a connection vertex and the first edge of $C$ is a terminal edge of one of the components of $G(\mathcal{F}_C \cap \mathcal{F}_H)$. The template also determines the sequence $u = (u_1, \ldots, u_b)$ of “gap lengths”, namely, the number of edges between consecutive components of $G(\mathcal{F}_C \cap \mathcal{F}_H)$, around $C$. Here $u_j$ is the number of edges of $G(\mathcal{F}_C \setminus \mathcal{F}_H)$ between the $j$th and $(j+1)$th components of $G(\mathcal{F}_C \cap \mathcal{F}_H)$ (and $u_b$ is the number of edges between the last and the first component). In Figure 2, if we take $x_2$ to be the start-vertex and choose the clockwise direction, then the template for $\mathcal{F}_C$ is $(1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 0)$. The sequence of intersection lengths is $\ell = (1, 3, 1)$ and the sequence of gap lengths is $u = (2, 4, 1)$.

Conversely, suppose that $b \geq 1$ and let $\ell, u$ be two sequences of $b$ positive integers which together sum to $k$. Then $(\ell, u)$ determines a template $\theta$ for $\mathcal{F}_C$, with respect to the chosen start vertex and direction. (The first $\ell_1$ entries of $\theta$ are 1, the next $u_1$ symbols are zero, the next $\ell_2$ entries are 1, and so on.)

When $b = 0$, either $\mathcal{F}_C = \mathcal{F}_H$ with template $\theta = (1, 1, \ldots, 1)$, or $\mathcal{F}_C \cap \mathcal{F}_H = \emptyset$ with template $\theta = (0, 0, \ldots, 0)$. The case $b = 0$ will be considered separately in many of our arguments.

We let $c$ denote the number of intersection lengths which are at least two, so $b - c$ counts the number of intersection lengths equal to 1 (that is, the number of entries of $\ell$ which equal 1).

### 3.2 A common framework

Let $r, s \geq 3$ be fixed integers. Recall that $X_k$ denotes the number of loose $k$-cycles in $\mathcal{F}(n, r, s)$, for $k \geq 2$, and $X_1$ is the number of loops (parts containing more than one point from some cell). We now describe the common framework that we will use when calculating $\mathbb{E}(YX_k)$, with $k = O(1)$, and $\mathbb{E}(Y^2) = \mathbb{E}(YX_1)$. These calculations are presented in Sections 4 and 5, respectively. We can write

$$
\mathbb{E}(YX_k) = \sum_{(\mathcal{F}_H, \mathcal{F}_C)} \Pr(\mathcal{F}_C \cup \mathcal{F}_H \subseteq \mathcal{F}(n, r, s))
$$

where the sum is over all pairs $(\mathcal{F}_H, \mathcal{F}_C)$ of subpartitions (not necessarily disjoint) such that $G(\mathcal{F}_H)$ is a loose Hamilton cycle and $G(\mathcal{F}_C)$ is a loose $k$-cycle. We will perform the summation using the following steps.

**Step 1: Choose $\mathcal{F}_H$.**

We must choose a loose Hamilton cycle $H$, and count the number of ways to choose parts $\mathcal{F}_H$ to correspond to the edges of $H$. Recalling (2.1), the number of choices of $\mathcal{F}_H$ is given by

$$
\frac{n!}{2t((s-2)!)^t} r^n(r-1)^t = \frac{p(rn)}{p(rn-st)} \mathbb{E}Y,
$$

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using (2.6) and the arguments given in the proof of Lemma 2.1.

**Step 2: Choose a template for $F_C$.**
When $b \geq 1$, we choose a vector $\ell = (\ell_1, \ldots, \ell_b)$ and a vector $u = (u_1, \ldots, u_b)$ of intersection lengths and gap lengths, respectively. This uniquely determines the corresponding template $\theta$, and also determines the parameters $a$ and $c$; that is, the number of parts in $F_C \setminus F_H$, and the number of intersection paths of length at least two, respectively. Denote the overall number of choices of templates for given values of $(a,b,c)$ by $M_2(k,a,b,c)$. When $k = O(1)$ we calculate $M_2(k,a,b,c)$ in Section 4, while the case that $k = t$ is analysed in Section 5.

When $b = 0$ there are two valid cases. We have $M_2(k,k,0,0) = 1$, since this corresponds to the case that $F_C \cap F_H = \emptyset$, which has the unique template $\theta = (0,0,\ldots,0)$. Similarly,

$$M_2(k,0,0,0) = \begin{cases} 1 & \text{if } k = t, \\ 0 & \text{otherwise}, \end{cases}$$

since this case can only arise when $k = t$ and $F_H = F_C$, corresponding to the unique template $\theta = (1,1,\ldots,1)$.

**Step 3: Identify $F_H \cap F_C$ and order components.**
First suppose that $b \geq 1$. We choose a permutation $\sigma$ of $[b]$ and select a sequence of $b$ vertex-disjoint induced subhypergraphs (paths) from $H$ with lengths $\ell_{\sigma(1)}, \ldots, \ell_{\sigma(b)}$ in order around $H$. The $j$'th component around $H$ becomes the $\sigma(j)$'th component around $C$ (with respect to the fixed start vertex and direction on $C$), matching the template for $F_C$. This determines the parts of $F_C \cap F_H$, together with an ordering of the components of $G(F_C \cap F_H)$ around $C$, as a sequence. In Lemma 3.2 below, we prove that there are

$$t \frac{(t-k+a-1)!}{(t-k+a-b)!}$$

ways to do this. (The orientation of these components is performed in Step 4.) If $b = 0$ then there is nothing to do in this step.

**Step 4: Choose the connection vertices and corresponding points.**
Now we determine the identity of the connection vertices, choose a point corresponding to each of these vertices and determine the orientation of each component of $G(F_C \cap F_H)$ within $C$. We prove in Lemma 3.3 below that there are

$$(2h(r,s))^b \left( \frac{(rs - r - s)^2}{h(r,s)} \right)^c$$

(3.2)
ways to do this, where
\[ h(r, s) = (r - 2)^2 + 2(s - 2)(r - 1)(r - 2) + \frac{1}{2}(s - 2)(s - 3)(r - 1)^2. \] (3.3)

Observe that \( h(r, s) > 0 \) whenever \( r, s \geq 3 \).

Again, if \( b = 0 \) then there is nothing to do in this step, and (3.2) gives the factor 1.

**Step 5: Choose the rest of \( F_C \setminus F_H \) and adjust for overcounting.**

Using the template \( \theta \), we must identify all vertices in edges of \( G(F_C \setminus F_H) \) other than the connection vertices, and assign points to these vertices, thereby completing all parts in \( F_C \setminus F_H \). Finally, we just adjust our counting so that each choice of \( F_C \) only arises once for a given choice of \( F_H \): we achieve this by dividing by the number of templates corresponding to \( F_C \). Let \( M_5(k, a, b) \) denote the number of ways to perform this step: it will turn out that this number is independent of \( c \).

We calculate \( M_5(k, a, b) \) in Section 4 when \( k = O(1) \), and in Section 5 when \( k = t \).

**Step 6: Multiply by the probability of containing \( F_C \cup F_H \).**

By now, all parts in \( F_C \cup F_H \) have been specified. So we multiply by the probability that the specified parts are contained in \( F(n, r, s) \), namely,
\[
\frac{p(rn - s(t + a))}{p(rn)} = \frac{p(rn - s(t + a))}{p(rn - st)} \cdot \frac{p(rn - st)}{p(rn)}.
\]

We express the probability in this form to display the inverse of a factor which arises in Step 1.

The case \( k = 1 \) is somewhat special, since \( X_1 \) counts all 1-cycles, not just loose 1-cycles. This will be discussed further in Section 4.

Combining Steps 1 to 6 leads to the following expression, which holds when \( r, s \geq 3 \):
\[
\frac{\mathbb{E}(YX_k)}{\mathbb{E}Y} = M_5(k, 0, 0) + M_5(k, k, 0) \cdot \frac{p(rn - s(t + k))}{p(rn - st)}
+ \sum_{\substack{a \geq b \geq 1, \\ c \geq 0}} M_2(k, a, b, c) \cdot \frac{t(t - k + a - 1)!}{(t - k + a - b)!} \cdot (2h(r, s))^b \cdot \left( \frac{(rs - r - s)^2}{h(r, s)} \right)^c
\times M_5(k, a, b) \cdot \frac{p(rn - s(t + a))}{p(rn - st)}.
\] (3.4)

Before we prove (3.1) and (3.2), we state the following lemma which contains two useful combinatorial facts (proof omitted). We adopt the convention that \( \binom{a}{0} = 1 \).

**Lemma 3.1.** Let \( R, T \) be positive integers with \( R \leq T \), and let \( J \) be a nonnegative integer.
(i) The number of sequences of $R$ positive integers which sum to $T$ is
\[ \binom{T-1}{R-1}. \]

(ii) The number of sequences of $R$ positive integers which sum to $T$ and which contain precisely $J$ entries equal to 1 is
\[ \binom{R}{J} \binom{T-R-1}{R-J-1} \]
if $R < T$, and equals 1 if $R = T$ (in which case also $J = T$).

In the next two lemmas, $H$ is a fixed Hamilton cycle. First we calculate the number of ways to perform Step 3 when $b \geq 1$.

**Lemma 3.2.** Suppose that a template for $F_C$ has been fixed, with $b \geq 1$. The number of ways to fix a permutation $\sigma$ of $[b]$ and choose $b$ vertex-disjoint induced subhypergraphs (paths) of $H$ with lengths $\ell_{\sigma(1)}, \ldots, \ell_{\sigma(b)}$ in order around $H$ is given by (3.1).

**Proof.** Fix an $H$-external start-vertex $w$ and direction on $H$, in $2t$ ways. We will ensure that $w$ belongs to precisely one edge $e$ of $G(F_C \cap F_H)$, namely, the first edge around $H$ from $w$ in the chosen direction. Choose a permutation $\sigma$ of $[b]$, in $b!$ ways. Let $\ell$ denote the sequence of intersection lengths corresponding to the fixed template. To choose paths of length $\ell_{\sigma(1)}, \ldots, \ell_{\sigma(b)}$ in this order around $H$, it suffices to choose a sequence of (positive) integers $(g_1, \ldots, g_b)$, which will be the gap lengths around $H$ in order. That is, the first $\ell_{\sigma(1)}$ parts of $F_H$ (in the chosen direction, starting from $w$) will form a component of $G(F_C \cap F_H)$, and then the next $g_1$ parts of $F_H$ will belong to $F_H \setminus F_C$, and so on.

Let $k - a$ be the number of edges in $G(F_C \cap F_H)$, as determined by the template. Then the gap lengths around $H$ must add up to $t - k + a$. By Lemma 3.1(i), there are $\binom{t-k+a-1}{b-1}$ ways to select these gap lengths around $H$, which determine the $b$ paths around $H$ with the given lengths. Finally we divide by $2b$, the number of choices of start-vertex and direction on $H$ which lead to the same choice of edges in $G(F_C \cap F_H)$. Multiplying these factors together gives
\[ \frac{t \binom{t-k+a-1}{b-1}!}{(t-k+a-b)!}, \]
completing the proof.

Next we calculate the number of ways to perform Step 4 when $b \geq 1$.

**Lemma 3.3.** Let $r, s \geq 3$ be fixed integers. Suppose that a template for $F_C$ has been fixed, with $b \geq 1$. The number of ways to select the $2b$ connection vertices, to assign a point to each, and to orient each component of $G(F_C \cap F_H)$ within $C$ is given by (3.2).
Proof. Each connection vertex is incident with one edge of $G(F_C \cap F_H)$, which we denote by $e$, and one edge of $G(F_C \setminus F_H)$, which we denote by $\hat{e}$. We must choose the connection vertex and assign a point to it in the part corresponding to the edge $\hat{e}$. Firstly, suppose that $v$ is a connection vertex in a terminal edge $e$ which belongs to a component of $G(F_C \cap F_H)$ of length at least 2. If $v$ equals the $H$-external vertex in $e$ then there is 1 choice for the vertex, and $r - 2$ ways to select a point corresponding to this vertex (in the part corresponding to $\hat{e}$). Otherwise, there are $s - 2$ $H$-internal vertices which can be chosen for $v$, and $r - 1$ ways to assign a point corresponding to this vertex. Overall, this gives $r - 2 + (s - 2)(r - 1) = rs - r - s$ ways to choose the connection vertex $v$ and a point corresponding to $v$ in the edge $\hat{e}$. The choice of $v$ has no effect on the number of choices for the connection vertex in the other terminal edge of this component, so we can simply square this contribution to take both connection vertices into account, giving a contribution of $(rs - r - s)^2$ in this case.

It remains to consider connection vertices which belong to 1-components of $G(F_C \cap F_H)$. The two connection vertices in $e$ may both be $C$-external, giving 1 choice (for the unordered pair of connection vertices) and $(r - 2)^2$ ways to assign the corresponding points. There are $2(s - 2)(r - 1)(r - 2)$ choices if one connection vertex in $e$ is $H$-external and the other is $H$-internal. (For example, see the edge containing vertices $z_2$ and $x_2$ in Figure 2.) Finally, if both connection vertices in $e$ are $H$-internal then there are $\frac{1}{2}(s - 2)(s - 3)(r - 1)^2$ choices for the connection vertices (as an unordered pair) and the corresponding points. (See the edge containing vertices $z_3$ and $z_4$ in Figure 2.) So the contribution in the second case is $h(r, s)$, as defined in (3.3).

Now that all connection vertices and their points have been identified, we decide the orientation of each component of $G(F_C \cap F_H)$ within $C$ by ordering the two connection vertices within each component. That is, we decide which connection vertex within a given component should be the “first” one we meet as we move around $C$ in the specified direction. There are $2^b$ choices of orientation.

Overall, the number of ways to select the $2b$ connection vertices, to assign a point to each, and to orient each component of $G(F_C \cap F_H)$ within $C$ is

$$2^b (rs - r - s)^{2c} h(r, s)^{b-c} = (2h(r, s))^b \left( \frac{(rs - r - s)^2}{h(r, s)} \right)^c,$$

as required. \hfill \Box

To apply (3.4), it remains to calculate $M_2(k, a, b, c)$ and $M_5(k, a, b)$ and perform the summation, in the two extreme cases, namely when $k = O(1)$ (in Section 4) and $k = t$ (in Section 5). Several simplifications make the calculations easier when $k$ is constant, allowing the use of generating functions to assist us with Steps 2 and 4. When $k = t$ we use Laplace summation to calculate the sum over all templates. This will involve detailed analysis of a certain real function of four variables.
4 Effect of short cycles

We use a generating function to perform Step 2 for short cycles.

**Lemma 4.1.** Suppose that $k \geq 1$ is fixed. If $a, b \geq 1$ and $c \geq 0$ then number of templates for $F_C$ (Step 2) with parameters $(a, b, c)$ is

$$M_2(k, a, b, c) = [x^k y^a z^b w^c] \left( \frac{x^2 y z (1 - x + x w)}{(1 - x)(1 - x y)} \right)^b.$$ 

**Proof.** We construct a generating function with the following variables:

- $x$ marks the number of edges of $C$,
- $y$ marks the number of parts in $F_C \setminus F_H$,
- $z$ marks the number of components in $G(F_C \cap F_H)$,
- $w$ marks the number of components in $G(F_C \cap F_H)$ of length at least 2.

Since $b \geq 1$, the template $\theta$ starts with some positive number of entries equal to 1, marking off the length of the first component of $G(F_C \cap F_H)$. If the first component has length 1 then it contributes one edge to $C$ and it contributes one component, so it is stored as $xz$. Otherwise, the first component gives one component, which is of length at least 2 and consists of $j + 2$ edges, for some $j \geq 0$. We store this in the generating function as $x^2 z w / (1 - x)$. Therefore the contribution of the first component of $F_C \cap F_H$ to the generating function is

$$xz + \frac{x^2 zw}{1 - x} = \frac{xz(1 - x + xw)}{1 - x}.$$ 

Next we must specify the first gap length, which must be at least 1. If the gap length is $j$ then this gap contributes $j$ edges to $C$ and $j$ edges to to $G(F_C \setminus F_H)$, so we store this in the generating function as

$$\frac{xy}{1 - xy}.$$ 

To complete the template we repeat the above procedure $b$ times in total. Hence when $b \geq 1$ we have

$$M_2(k, a, b, c) = [x^k y^a z^b w^c] \left( \frac{x^2 y z (1 - x + x w)}{(1 - x)(1 - x y)} \right)^b.$$ 

$\square$
Next we perform Step 5. Recall that during Steps 1–4 we have identified \( F_H \), the template \( \theta \), the parts in \( F_C \cap F_H \) and the connection vertices. In Step 5 we determine the rest of \( F_C \setminus F_H \) and adjust for overcounting, usually by dividing by the number of templates corresponding to a given \( F_C \).

**Lemma 4.2.** Let \( k \geq 1 \) be a fixed integer and let \( a, b, c \) be nonnegative integers with \( c \leq b \leq k - a \), such that if \( b = 0 \) then \( a = k \). If \( b \geq 1 \) then \( M_5(k, a, b) \) is asymptotically equal to

\[
\frac{1}{2b} \left( \frac{(r-2)(rs-r-s-1)(rs-r-s)^{s-2}t^{s-1}}{(s-2)!} \right)^a ((r-2)(rs-r-s-1)t)^{-b},
\]

while \( M_5(k, k, 0) \) is asymptotically equal to

\[
\frac{1}{2k} \left( \frac{(r-2)(rs-r-s-1)(rs-r-s)^{s-2}t^{s-1}}{(s-2)!} \right)^k.
\]

Finally, \( M_5(k, 0, 0) = 0 \).

**Proof.** The last statement is clear since the case \( a = 0 \) can only arise when \( k = t \) and \( F_H = F_C \), so this case is ruled out by the assumption that \( k = O(1) \).

Next, consider the case that \( k = 1 \). The random variable \( X_1 \) counts all 1-cycles, not just loose 1-cycles. But loose 1-cycles involve \( s - 1 \) distinct cells, while non-loose 1-cycles involve at most \( s - 2 \) distinct cells. This implies that the contribution to \( M_5(1, 1, 0) \) from non-loose 1-cycles is \( O(1/n) \) times the contribution to \( M_5(1, 1, 0) \) from loose 1-cycles only. Hence when \( k = 1 \), it suffices to only consider loose 1-cycles \((a = k = 1 \text{ and } b = 0)\). As \( X_k \) counts the number of loose \( k \)-cycles, when \( k \geq 2 \), this brings the \( k = 1 \) case in line with the general case, so we may treat them both together below.

Let \( \theta \) be the template which was chosen for \( F_C \) in Step 2. We must identify all \( C \)-external vertices which are not incident with an edge of \( G(F_C \cap F_H) \) (there are \( a - b \) of them), and all \( C \)-internal vertices which are not incident with an edge of \( G(F_C \cap F_H) \) (there are \((s-2)a \) of them). We call the vertices identified in this step the new vertices. This will specify the identity of all remaining vertices in \( G(F_C \setminus F_H) \). The template \( \theta \) determines a fixed start-vertex and direction around \( C \). Since \( k \) is constant, as we move around the cycle \( C \) identifying vertices, there are always \( t - O(1) \sim t \) remaining \( H \)-external vertices to choose from, and there are always \( n - t - O(1) \sim (s-2)t \) remaining \( H \)-internal vertices to choose from. For a vertex \( v \) which we have just identified, the number of choices for points representing \( v \) in parts corresponding to the edges of \( G(F_C \setminus F_H) \) incident with \( v \) is

\[
\begin{cases}
(r-2)(r-3) & \text{if } v \text{ is } C\text{-external and } H\text{-external}, \\
(r-1)(r-2) & \text{if } v \text{ is } C\text{-external and } H\text{-internal}, \\
(r-2) & \text{if } v \text{ is } C\text{-internal and } H\text{-external}, \\
(r-1) & \text{if } v \text{ is } C\text{-internal and } H\text{-internal}.
\end{cases}
\]
This gives a sequence of $C$-internal vertices for each edge, but we need a set. If an edge in $G(F_C \setminus F_H)$ has $j$ $C$-internal vertices which are $H$-external, then we must divide by $j!(s - 2 - j)! = (s - 2)!/(s-j)^{2}!$.

Hence the number of ways to identify the new $C$-external vertices of $G(F_C \setminus F_H)$, and assign points to them, is asymptotically equal to

$$\sum_{d=0}^{a-b} \binom{a-b}{d} ((r-2)(r-3)t)^d ((r-1)(r-2)(s-2)t)^{a-b-d} = ((r-2)(rs-r-s-1)t)^{a-b},$$

and the number of ways to identify the $C$-internal vertices of $G(F_C \setminus F_H)$, and assign points to them, is asymptotically equal to

$$\left(\frac{1}{(s-2)!}\sum_{j=0}^{s-2} \binom{s-2}{j} ((r-2)t)^j ((r-1)(s-2)t)^{s-2-j}\right)^a = \left(\frac{(rs-r-s)^{s-2}t^{s-2}}{(s-2)!}\right)^a.$$

If $b \geq 1$ then each choice of $F_C$ corresponds to precisely $2b$ templates. So the expression for $M_5(k,a,b)$ follows by multiplying (4.1) and (4.2) and dividing by $2^b$.

If $b = 0$ then $a = k$ and the argument above determines a sequence of parts of $F_C$, with respect to some given start-vertex and direction. We must divide by $2k$ to adjust for this multiple counting. Indeed, the choice of direction really corresponds to the choice of one of the two points representing the start vertex $v$ in $C$. So we must also divide by 2 when $k = 1$, to replace the ordered pair of points representing $v$ by an unordered pair of points. This completes the proof.\(\square\)

We now have all the information we need in order to perform the summation in (3.4).

**Lemma 4.3.** Let $r, s \geq 3$ be fixed integers. For any fixed integer $k \geq 1$,

$$\frac{\mathbb{E}(YX_k)}{\mathbb{E}Y} \sim \frac{(r-1)(s-1))^k}{2k} + \frac{\zeta_1^k}{2k} + \frac{\zeta_2^k}{2k} - \frac{1}{2k}$$

as $n \to \infty$ along $\mathcal{I}(r,s)$, where $\zeta_1, \zeta_2 \in \mathbb{C}$ satisfy

$$\zeta_1 + \zeta_2 = -\frac{rs^2 - s^2 - 2rs + r + 2}{rs - r - s}, \quad \zeta_1\zeta_2 = \frac{(s-1)(s-2)(r-1)}{rs - r - s}.$$  \(4.3\)

**Proof.** Fix $k \geq 1$. Before applying Lemmas 4.1 and 4.2, we simplify some factors of (3.4). Since $k - a = O(1)$, the factor of (3.4) from Step 3 equals

$$\frac{t(t - k + a - 1)!}{(t - k + a - b)!} \sim t^b.$$
Similarly, using Stirling’s formula, the factor of (3.4) from Step 6 equals

\[ p(rn-st-sa) \sim \left( \frac{(s-1)!}{(rs-r-s)^{s^{-1}ts^{-1}}} \right)^a. \]

Combining these with Lemmas 4.1 and 4.2, the expression (3.4) becomes

\[
\frac{\mathbb{E}(YX_k)}{\mathbb{E}^2} \sim \frac{\mu_k^k}{2k} + \sum_{a \geq b \geq 1, \ c \geq 0} [x^k y^a z^b w^c] \frac{1}{2b} \left( \frac{x^2yz(1-x+xw)}{(1-x)(1-xy)} \right)^b \mu_1^a \mu_2^b \mu_3^c
\]

\[
= \frac{\mu_k^k}{2k} - \frac{1}{2} \sum_{a,c \geq 0} [x^k y^a w^c] \ln \left( 1 - \frac{\mu_1 \mu_2 x^2 y(1-x + \mu_3 xw)}{(1-x)(1-\mu_1 xy)} \right),
\]

where

\[
\begin{align*}
\mu_1 &= \frac{(s-1)(r-2)(rs-r-s-1)}{rs-r-s} \\
\mu_2 &= 2 \frac{h(r,s)}{(r-2)(rs-r-s-1)} \\
\mu_3 &= \frac{(rs-r-s)^2}{h(r,s)}.
\end{align*}
\]

(4.4)

Observe that \( \mu_1, \mu_2, \mu_3 \) are well-defined when \( r, s \geq 3 \). The summation over \( a \) and \( c \) can be achieved by setting \( y = w = 1 \), giving

\[
\frac{\mathbb{E}(YX_k)}{\mathbb{E}^2} \sim \frac{\mu_k^k}{2k} - \frac{1}{2} [x^k] \ln \left( 1 - \frac{\mu_1 \mu_2 x^2 (1 - (1 - \mu_3) x)}{(1-x)(1-\mu_1 x)} \right)
\]

\[
= \frac{1}{2} [x^k] \ln \left( \frac{1 - (\mu_1 + 1)x - \mu_1 (\mu_2 - 1)x^2 - \mu_1 \mu_2 (\mu_3 - 1)x^3}{1-x} \right)
\]

\[
= \frac{1}{2} [x^k] \ln \left( \frac{(1 - (r-1)(s-1)x) \left( 1 + \frac{rs^2 - s^2 - 2rs + r + 2}{rs - r - s} x + \frac{(s-1)(s-2)(r-1)x^2}{rs - r - s} \right)}{1-x} \right)
\]

using (4.4) for the final equality. The quadratic factor inside the logarithm factors as

\[
1 + \frac{rs^2 - s^2 - 2rs + r + 2}{rs - r - s} x + \frac{(s-1)(s-2)(r-1)x^2}{rs - r - s} = (1 - \zeta_1 x)(1 - \zeta_2 x)
\]

where the roots \( \zeta_1, \zeta_2 \in \mathbb{C} \) are defined by (4.3). Using this factorisation we can write

\[
\frac{\mathbb{E}(YX_k)}{\mathbb{E}^2} \sim \frac{1}{2} [x^k] \left( \ln (1 - (r-1)(s-1)x) + \ln(1 - \zeta_1 x) + \ln(1 - \zeta_2 x) - \ln(1-x) \right)
\]

\[
= \frac{(r-1)(s-1)^k}{2k} + \frac{\zeta_1^k}{2k} + \frac{\zeta_2^k}{2k} - \frac{1}{2k},
\]

as claimed. \( \square \)
The following corollary follows from Lemma 2.5, (2.8), (2.9) and Lemma 4.3.

**Corollary 4.4.** Suppose that $r, s \geq 3$ are fixed integers. Then condition (A2) of Theorem 2.4 holds with $\lambda_k$ given by (2.8) and $\delta_k$ defined by

$$\delta_k = \frac{\zeta_1^k + \zeta_2^k - 1}{((r - 1)(s - 1))^k}$$

for $k \geq 1$.

Observe that even though $\zeta_1, \zeta_2$ may be complex, $\delta_k$ is always real, by de Moivre’s Theorem.

Now we can complete the proof of Theorem 1.2, giving the expected number of loose Hamilton cycles in a random $r$-regular $s$-uniform hypergraph when $r, s \geq 2$ are fixed integers and $(r, s) \neq (2, 2)$.

**Proof of Theorem 1.2.** Janson [11, Theorem 2] proved Theorem 1.2 holds when $s = 2$ and $r \geq 3$, and Lemma 2.3 states that the result holds when $(r, s) = (2, 3)$. Now assume that $r, s \geq 3$. The hypergraph $G(F)$ is simple if and only if $F$ has no 1-cycles (that is, $X_1 = 0$) and no repeated edges. The probability that two parts of $F(n, r, s)$ give rise to a repeated edge is

$$O\left(\frac{n^s p(rn - 2s)}{p(rn)}\right) = O(n^{2-s}) = o(1)$$

when $s \geq 3$. In other words, a.a.s. $G(F)$ has no repeated edges and it suffices to condition on $X_1 = 0$. Corollary 4.4 states that condition (A2) of Theorem 2.4 holds, which implies that

$$\frac{\mathbb{E}Y_G}{\mathbb{E}Y} \sim \frac{\mathbb{E}(Y | X_1 = 0)}{\mathbb{E}Y} \rightarrow e^{-\lambda_1 \delta_1}.$$ 

Using (2.8), (4.3) and Corollary 4.4 we have

$$-\lambda_1 \delta_1 = \frac{1 - \zeta_1 + \zeta_2}{2} = \frac{(s - 1)(rs - s - 2)}{2(rs - r - s)},$$

and combining this with Corollary 2.2 completes the proof. \qed

### 4.1 Preparation for small subgraph conditioning

Before proceeding to the second moment calculations, we establish some results which we will be needed in order to apply Theorem 2.4. Recall the definition of $\zeta_1, \zeta_2$ from (4.3), and the definition of $\delta_k$ from (4.5).

The first result shows that $\delta_k > -1$ for all $k \geq 1$. This will be needed in the proof of the threshold result, Theorem 1.1.

**Lemma 4.5.** Suppose that $r, s \geq 3$. Then $\delta_k > -1$ for all $k \geq 1$. 

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Proof. For ease of notation, write \( A = (r-1)(s-1) \). The desired inequality is equivalent to
\[
A^k + \zeta_1^k + \zeta_2^k - 1 > 0.
\] (4.7)

First suppose that \( \zeta_1 \) and \( \zeta_2 \) are not real. In this case, \( \zeta_1 \) and \( \zeta_2 \) form a complex conjugate pair, and
\[
A^k + \zeta_1^k + \zeta_2^k - 1 \geq A^k - |\zeta_1 \zeta_2|^{k/2} - 1
\]
using (4.3). This expression is positive for all \( r, s \geq 3 \) and \( k \geq 1 \).

For the remainder of the proof, suppose that \( \zeta_1 \) and \( \zeta_2 \) are real. Then (4.7) holds for all even \( k \geq 2 \). We will prove by induction that
\[
|\zeta_1|^k + |\zeta_2|^k < A^k - 1
\] (4.8)
for all \( k \geq 1 \). Since \( \zeta_1 \) and \( \zeta_2 \) are both negative, by (4.3), we see that (4.7) and (4.8) are equivalent when \( k \) is odd.

Firstly, note that (4.8) holds when \( k = 1 \), using (4.3) and the fact that \( r, s \geq 3 \). Now suppose that \( k \geq 2 \), and observe that
\[
\zeta_1^k + \zeta_2^k = (\zeta_1 + \zeta_2) (\zeta_1^{k-1} + \zeta_2^{k-1}) - \zeta_1 \zeta_2 (\zeta_1^{k-2} + \zeta_2^{k-2}).
\]
The terms \( \zeta_1^k + \zeta_2^k \), \( (\zeta_1 + \zeta_2) (\zeta_1^{k-1} + \zeta_2^{k-1}) \) and \( \zeta_1 \zeta_2 (\zeta_1^{k-2} + \zeta_2^{k-2}) \) have the same sign, since \( \zeta_1 \) and \( \zeta_2 \) are both negative. Therefore, by the inductive hypothesis,
\[
|\zeta_1|^k + |\zeta_2|^k = |\zeta_1^k + \zeta_2^k| \leq |(\zeta_1 + \zeta_2) (\zeta_1^{k-1} + \zeta_2^{k-1})|
= (|\zeta_1| + |\zeta_2|) \left( |\zeta_1|^{k-1} + |\zeta_2|^{k-1} \right)
< (A - 1)(A^{k-1} - 1)
\leq A^k - 1,
\]
since \( A^{k-1} + A \geq 2 \) when \( r, s \geq 3 \). This completes the inductive step.

Next, we show that condition (A3) of Theorem 2.4 holds, under weak conditions on \( r \) and \( s \).

**Lemma 4.6.** Let \( s \geq 3 \) and \( r \geq s + 1 \). With \( \lambda_k \) as defined in (2.8) and \( \delta_k \) as defined in (4.5), we have
\[
\exp \left( \sum_{k \geq 1} \lambda_k \delta_k^2 \right) = \frac{r (rs - r - s)}{(r - 2) \sqrt{Q(r, s)}}
\] (4.9)
where
\[
Q(r, s) = r^2 s^2 - rs^3 - 2r^2s + 3rs^2 + s^3 + r^2 - 6rs + 4r - 4s + 4.
\]
Furthermore, \( Q(r, s) > 0 \) so condition (A3) of Theorem 2.4 holds.
Proof. Again we write $A = (r - 1)(s - 1)$ for ease of notation. We calculate

$$
\sum_{k \geq 1} \lambda_k \delta_k^2 = \sum_{k \geq 1} \frac{1}{2k \cdot A^k} (\zeta_1^k + \zeta_2^k - 1)^2
$$

$$
= \sum_{k \geq 1} \frac{1}{2k \cdot A^k} \left( \zeta_1^{2k} + \zeta_2^{2k} + 2(\zeta_1 \zeta_2)^k - 2\zeta_1^k - 2\zeta_2^k + 1 \right)
$$

$$
= -\frac{1}{2} \ln \left(1 - \frac{\zeta_2^2}{A}\right) - \frac{1}{2} \ln \left(1 - \frac{\zeta_1^2}{A}\right) - \ln \left(1 - \frac{\zeta_1 \zeta_2}{A}\right) + \ln \left(1 - \zeta_1/A\right) + \ln \left(1 - \zeta_2/A\right) - \frac{1}{2} \ln \left(1 - \frac{1}{A}\right).
$$

Therefore

$$
\exp \left( \sum_{k \geq 1} \lambda_k \delta_k \right) = \left( \frac{A \cdot (A - \zeta_1)^2 \cdot (A - \zeta_2)^2}{(A - \zeta_1^2)(A - \zeta_2^2)(A - 1)(A - \zeta_1 \zeta_2)^2} \right)^{1/2}
$$

$$
= \left( \frac{A \cdot (A^4 - 2(\zeta_1 + \zeta_2)A^3 + ((\zeta_1 + \zeta_2)^2 + 2\zeta_1 \zeta_2)A^2 - 2\zeta_1 \zeta_2(\zeta_1 + \zeta_2)A + (\zeta_1 \zeta_2)^2)}{(A - 1)(A - \zeta_1 \zeta_2)^2(A^2 - ((\zeta_1 + \zeta_2)^2 - 2\zeta_1 \zeta_2)A + (\zeta_1 \zeta_2)^2)} \right)^{1/2}.
$$

Substituting for $A$ and for $\zeta_1 + \zeta_2$ and $\zeta_1 \zeta_2$ leads to (4.9) after much simplification, using (4.3).

For the final statement, rewrite $Q$ as

$$
Q(r, s) = (s - 1)^2 r^2 - (s - 1)(s^2 - 2s + 4)r + s^3 - 4s + 4.
$$

For fixed $s \geq 2$, the roots of this quadratic in $r$ occur at

$$
\frac{s^2 - 2s + 4 \pm s \sqrt{(s - 2)(s - 6)}}{2(s - 1)}.
$$

It follows that $Q(r, s)$ is positive whenever $r \geq \frac{s^2 - s + 2}{s - 1}$, and since $s \geq 3$ this inequality holds whenever $r \geq s + 1$.

\section{The second moment}

In this section we calculate the second moment of $Y$, under the assumptions that $s \geq 3$ and $r > \rho(s)$. We use the framework from Section 3, but write $F_1$ and $F_2$ rather than $F_H$ and $F_C$, respectively, and let $H_j = G(F_j)$ for $j = 1, 2$.

First we state the following combinatorial fact without proof.

\textbf{Lemma 5.1.} Let $J, R, T$ be positive integers and let

$$
\mathcal{J} = \left\{ (j_1, \ldots, j_R) \in \{0, 1, \ldots, J\}^R \left| \sum_{i=1}^R j_i = T \right. \right\}.
$$

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That is, \( J \) is the set of all nonnegative integer \( R \)-tuples which sum to \( T \) and with no entry greater than \( J \). Then

\[
\sum_{(j_1, \ldots, j_R) \in J} \prod_{i=1}^R \frac{1}{j_i! (J-j_i)!} = \frac{1}{(J!)^R} \binom{JR}{T}.
\]

We now give an expression for \( M_2(t, a, b, c) \), required for Step 2.

**Lemma 5.2.** Suppose that \( a, b, c, t \) are nonnegative integers with \( 0 \leq c \leq t - a - b \), such that if \( b = 0 \) then \( a = 0 \) or \( a = t \). When \( b \geq 1 \), the number of templates for \( F_2 \) with parameters \( (a, b, c) \) is

\[
M_2(t, a, b, c) = b \xi_t(a, b, c) \binom{a}{b} \binom{b}{c} \binom{t-a-b}{c},
\]

(5.1)

where \( \xi_t(a, b, c) \) is defined by

\[
\xi_t(a, b, c) = \begin{cases} \frac{c}{t-a-b} & \text{if } a + b < t, \\ 1 & \text{if } a + b = t. \end{cases}
\]

(5.2)

When \( b = 0 \) we define \( M_2(t, 0, 0, 0) = M_2(t, t, 0, 0) = 1 \).

**Proof.** First, suppose that \( 1 \leq b \leq t - a - 1 \). By Lemma 3.1(ii), there are

\[
\binom{b}{c} \binom{t-a-b-1}{c-1} = \frac{c}{t-a-b} \binom{b}{c} \binom{t-a-b}{c}
\]

ways to select a sequence \( \ell = (\ell_1, \ldots, \ell_b) \) of intersection lengths which add to \( t - a \), such that precisely \( b - c \) of these lengths equal 1 and the rest are at least 2. Then Lemma 3.1(i), there are

\[
\binom{a-1}{b-1} = \frac{b}{a} \binom{a}{b}
\]

ways to choose a sequence \( u = (u_1, \ldots, u_b) \) of gap lengths around \( H_2 \). Multiplying these expressions together gives (5.1).

Next suppose that \( b = t - a \geq 0 \). The given lower and upper bounds on \( c \) imply that \( c = 0 \). Furthermore, we have \( t = a + b \leq 2a \). There is one way to choose the vector \( \ell \) of intersection lengths, and the number of choices for the sequence \( u \) of gap lengths is \( \frac{t-a}{a} \binom{a}{t-a} \), as above. This leads to the stated value for \( M_2(t, a, t-a, t-a) \), using (5.2) and recalling that \( \binom{0}{0} = 1 \).

Finally suppose that \( b = 0 \) (and hence \( c = 0 \)). There are only two possibilities: \( a = 0 \) and \( F_1 = F_2 \), corresponding to the unique template \( \theta = (1, 1, \ldots, 1) \), or \( a = t \) and \( F_1 \cap F_2 = \emptyset \), corresponding to the unique template \( \theta = (0, 0, \ldots, 0) \). This matches the values \( M_2(t, 0, 0, 0) = M_2(t, t, 0, 0) = 1 \). □
Next we perform Step 5. Recall that during Steps 1–4 we have identified $F_1$, the template $\theta$, the parts in $F_1 \cap F_2$ and the connection vertices. In Step 5 we determine the rest of $F_2 \setminus F_1$ and adjust for overcounting, usually by dividing by the number of templates corresponding to a given $F_2$.

**Lemma 5.3.** Let $a, b, t$ be integers which satisfy $0 \leq b \leq a \leq t$. If $b \geq 1$ then

$$M_5(t, a, b) = \frac{1}{2b} (a - b)! ((s - 2)a)! \left( \frac{(r - 1)^{s - 2} (r - 2)^2}{(s - 2)!} \right)^a (r - 2)^{-2b} \sum_{d=0}^{a-b} \binom{a-b}{d} \binom{(s-2)a}{a-b-d} \binom{r-3}{r-2}^d,$$

while if $b = 0$ and $a = t$ then

$$M_5(t, t, 0) = \frac{1}{2t} t! ((s-2)t)! \left( \frac{(r - 1)^{s - 2} (r - 2)^2}{(s - 2)!} \right)^t \sum_{d=0}^{t} \binom{t}{d} \binom{(s-2)t}{t-d} \binom{r-3}{r-2}^d.$$

Finally, $M_5(t, 0, 0) = 1$.

**Proof.** Let $\theta$ be the template for $F_2$ which was fixed during Step 2. In Step 5, we must identify all $H_2$-external vertices in $G(F_2 \setminus F_1)$ which are not connection vertices (there are $a - b$ of them), and all $H_2$-internal vertices in $G(F_2 \setminus F_1)$ (there are $(s-2)a$ of them).

As in Lemma 4.2, we call all vertices identified in this step *new*. We must also assign points to all new vertices, thereby completing $F_2 \setminus F_1$. In Section 4 we approximated our number of choices at each step by $t$ or $(s - 2)t$, respectively, since we only had to identify a constant number of new vertices. Here we must count more carefully, and we will need a new parameter.

Let $d$ be the number of new $H_2$-external vertices which are also $H_1$-external. Then there are $a - b - d$ new $H_2$-external vertices which are $H_1$-internal, and there are $a - b - d$ new $H_2$-internal vertices which are $H_2$-external. Finally, there are $(s-2)a - (a-b-d) = (s-3)a + b + d$ new $H_2$-internal vertices which are $H_1$-internal. We must select identities and points for all these new vertices.

To do this, first order all remaining $H_1$-external vertices (those not already present in $G(F_1 \cap F_2)$) and order all remaining $H_1$-internal vertices (those not already present in $G(F_1 \cap F_2)$), in

$$(a - b)! ((s - 2)a)!$$

ways. We will work around $H_2$ in the order specified by the template $\theta$. When we assign identities to new vertices of $H_2$, we will always take the first vertex from the appropriate list (either $H_1$-external or $H_1$-internal), and delete it from this list after it has been assigned. At the end of this process, both lists will be empty and all new vertices in $H_2$ will have been identified.
To begin, select which \( d \) new \( H_2 \)-external vertices will be \( H_1 \)-external, and assign identities to these vertices in order, from the ordered list of remaining \( H_1 \)-external vertices. This can be done in

\[
\binom{a-b}{d}
\]  

ways, leaving \( a-b-d \) new \( H_2 \)-external vertices to be identified. These \( a-b-d \) new \( H_2 \)-external vertices must be \( H_1 \)-internal, and we move around \( H_2 \) in order, assigning identities to these vertices from the ordered list of \( H_1 \)-internal vertices. (There is only one way to do this.)

Next, by Lemma 5.1, there are

\[
\sum_{(j_1, \ldots, j_a) \in J} \prod_{\ell=1}^{a} \frac{1}{j_\ell! (s-2-j_\ell)!} = \frac{1}{((s-2)!)^a} \binom{(s-2)a}{a-b-d}
\]  

ways to decide how many new \( H_2 \)-internal vertices in each edge of \( G(F_2 \setminus F_1) \) will be \( H_1 \)-external: let the \( i \)th such edge contain \( j_i \) \( H_2 \)-internal vertices which are \( H_1 \)-external. (Here \( J \) is the set of all nonnegative integer sequences of length \( a \) which add up to \( a-b-d \), such that no entry is greater than \( s-2 \).) We visit each edge of \( G(F_2 \setminus F_1) \) in order, and in the \( i \)th such edge we assign identities to \( j_i \) new \( H_2 \)-internal vertices (namely, the first \( j_i \) remaining elements from the ordered list of \( H_1 \)-external vertices), and we assign identities to the \( s-2-j_i \) other internal vertices of this edge (from the ordered list of remaining \( H_1 \)-internal vertices). We must divide by the factorials to adjust for symmetry, since the \( H_2 \)-internal vertices within a new edge should form a set, not a sequence.

This gives every new vertex of \( H_2 \) an identity, and now we must assign points. The \( d \) new \( H_2 \)-external vertices which are \( H_1 \)-external and the \( a-b-d \) new \( H_2 \)-external vertices which are \( H_1 \)-internal must all be assigned precisely two points, and all remaining new vertices must be assigned precisely one point. There are

\[
((r-2)(r-3))^d ((r-1)(r-2))^{a-b-d} (r-2)^{a-b-d} (r-1)^{(s-3)a+b+d} = ((r-1)^{s-2}(r-2)^2)^a (r-2)^{-2b} \left( \frac{r-3}{r-2} \right)^d
\]  

ways to assign points to these new vertices (in the parts corresponding to the edges of \( G(F_2 \setminus F_1) \)).

If \( b \geq 1 \) then \( F_2 \) arises from precisely 2\( b \) templates, so we multiply (5.3)–(5.6) together and divide by \( 2^b \), giving the desired expression. If \( b = 0 \) and \( a = t \) then the above argument determines a sequence of parts of \( F_2 \), with respect to some given start-vertex and direction. We must divide by \( 2t \) to adjust for this multiple counting. This leads to the stated expression for \( M_5(t,t,0) \). Finally, if \( a = b = 0 \) then \( F_2 = F_1 \) and there is nothing to do in this step, so \( M_5(t,0,0) = 1 \) as claimed.
Define
\[ D = \{(a, b, c, d) \in \mathbb{Z}^4 \mid 0 \leq c \leq b, \ 0 \leq d \leq a - b, \ a + b + c \leq t\} \]
and
\[ \hat{D} = D \setminus \{(a, 0, 0, d) \in D \mid 1 \leq a \leq t - 1\}. \]
The set \( \hat{D} \) contains all possible 4-tuples of parameters which can arise in the second moment calculation, recalling that when \( b = 0 \) we must have \( a = 0 \) or \( a = t \), for combinatorial reasons.

The next lemma finds a combinatorial expression for \( \mathbb{E}(Y^2)/\left(\mathbb{E}Y\right)^2 \) as a summation over \( \hat{D} \), with the summands defined below. However, it will prove easier to calculate the sum over the slightly larger set \( D \). As we will see, the additional terms will have only negligible effect on the answer. Hence we define the summand \( J_t(a, b, c, d) \) for all \((a, b, c, d) \in D\), as follows. First, let
\[
\begin{aligned}
\kappa_2 &= \frac{2h(r, s)}{(r - 2)^2} \\
\kappa_3 &= \frac{(rs - r - s)^2}{h(r, s)} \\
\kappa_4 &= \frac{r - 3}{r - 2}
\end{aligned}
\] (5.7)
where, as defined in (3.3),
\[ h(r, s) = (r - 2)^2 + 2(s - 2)(r - 1)(r - 2) + \frac{1}{2}(s - 2)(s - 3)(r - 1)^2. \]
Observe that \( \kappa_2, \kappa_3, \kappa_4 \) are well-defined whenever \( s \geq 3 \) and \( r > \rho(s) \).

**Lemma 5.4.** Suppose that \( s \geq 3 \) and \( r > \rho(s) \) are fixed integers. Then
\[
\frac{\mathbb{E}(Y^2)}{(\mathbb{E}Y)^2} = \sum_{(a,b,c,d) \in \hat{D}} J_t(a, b, c, d)
\]
where the summands are defined as follows:

- If \( a = 0 \) then \( b = c = d = 0 \) and we define \( J_t(0, 0, 0, 0) = \frac{1}{\mathbb{E}Y} \).
- If \( a \geq 1 \) then we let
  \[
  J_t(a, b, c, d) = \frac{\xi_t(a, b, c) t}{2a^2} \frac{a}{b} \frac{b}{c} \frac{(t - a - b)}{c} a! ((s - 2)a)! \\
  \times \left( \frac{a - b}{d} \right) \left( \frac{(s - 2)a}{a - b - d} \right) \left( \frac{(r - 1)^{s-2} (r - 2)^2}{(s - 2)!} \right)^a \frac{\kappa_2^b \kappa_3^c \kappa_4^d}{p(rn - s(t + a))} \frac{1}{p(rn - st)} \frac{1}{\mathbb{E}Y},
  \]
where \( \xi_t(a, b, c) \) is defined in (5.2).
Proof. The set $\hat{D}$ contains all values of the parameters $(a, b, c, d)$ which can arise from the interaction of two loose Hamilton cycles. After dividing (3.4) by $\mathbb{E}Y$, we can write the resulting expression as a sum over $\hat{D}$, and denote the summand corresponding to $(a, b, c, d) \in \hat{D}$ by $J_t(a, b, c, d)$. (Recall that the sum over $d$ arises in the factor $M_5(t, a, b)$, see Lemma 5.3.)

When $b \geq 1$, substituting Lemma 5.2 and Lemma 5.3 into (3.4) and dividing by $\mathbb{E}Y$ shows that the summand $J_t(a, b, c, d)$ equals the expression given in above.

When $a = 0$ we have $b = c = d = 0$, corresponding to the term 
$$\frac{M_5(t, 0, 0)}{\mathbb{E}Y} = \frac{1}{\mathbb{E}Y}.$$ 
This equals the definition of $J_t(0, 0, 0, 0)$ given above. Finally, suppose that $a = t$ and $b = c = 0$, which corresponds to 
$$\frac{M_5(t, t, 0)}{\mathbb{E}Y} \cdot \frac{p(rn - st)}{p(rn - st)} = \frac{t! ((s - 2)t)!}{2t} \left( \frac{(r - 1)^{(s-2)(r-2)^2}}{(s-2)!} \right)^t \sum_{d=0}^{t} \binom{t}{d} \binom{(s-2)t}{t-d} \times \left( \frac{r - 3}{r - 2} \right)^d \frac{p(rn - 2st)}{p(rn - st)} \frac{1}{\mathbb{E}Y}.$$ 
This expression equals $\sum_{d=0}^{t} J_t(t, 0, 0, d)$, with $J_t(t, 0, 0, d)$ as defined above, noting that $\xi_t(t, 0, 0) = 1$. \hfill $\square$

The summation in Lemma 5.4 will be evaluated using Laplace summation. The following lemma is tailored for this purpose: it is a restatement of [9, Lemma 6.3] (using the notation of the current paper).

Lemma 5.5. Suppose the following:

(i) $\mathcal{L} \subset \mathbb{R}^m$ is a lattice with full rank $m$.

(ii) $K \subset \mathbb{R}^m$ is a compact convex set with non-empty interior $K^\circ$.

(iii) $\varphi : K \to \mathbb{R}$ is a continuous function with a unique maximum at some interior point $x^* \in K^\circ$.

(iv) $\varphi$ is twice continuously differentiable in a neighbourhood of $x^*$ and the Hessian $H^* := D^2 \varphi(x^*)$ is strictly negative definite.

(v) $\psi : K^* \to \mathbb{R}$ is a continuous function on some neighbourhood $K^* \subseteq K$ of $x^*$ with $\psi(x^*) > 0$.

(vi) For each positive integer $t$ there is a vector $w_t \in \mathbb{R}^m$. 

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(vii) For each positive integer \( t \) there is a function \( J_t : (\mathcal{L} + \mathbf{w}_t) \cap tK \to \mathbb{R} \) and a real number \( b_t > 0 \) such that, as \( t \to \infty \),

\[
J_t(v) = O \left( b_t e^{\psi(v/t) + o(t)} \right), \quad v \in (\mathcal{L} + \mathbf{w}_t) \cap tK,
\]

and

\[
J_t(v) = b_t (\psi(v/t) + o(1)) e^{t\varphi(v/t)}, \quad v \in (\mathcal{L} + \mathbf{w}_t) \cap tK^*,
\]

uniformly for \( v \) in the indicated sets.

Then, as \( t \to \infty \),

\[
\sum_{v \in (\mathcal{L} + \mathbf{w}_t) \cap tK} J_t(v) \sim \frac{(2\pi)^{m/2} \psi(x^*)}{\det(\mathcal{L}) \sqrt{\det(-H^*)}} b_t t^{m/2} e^{t\varphi(x^*)}.
\]

As remarked in [9], the result also holds if \( t \) tends to infinity along some infinite subset of the positive integers.

In order to apply Lemma 5.5, we need some more notation. Define the scaled domain

\[
K = \{ (\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4 \mid 0 \leq \gamma \leq \beta, \ 0 \leq \delta \leq \alpha - \beta, \ \alpha + \beta + \gamma \leq 1 \}.
\]

Observe that \( \mathcal{D} \) can be written as the intersection of \( \mathbb{Z}^4 \) with \( tK \), but it is not possible to write \( \hat{\mathcal{D}} \) in this form. This is the reason why it is more convenient to work with \( \mathcal{D} \) when performing Laplace summation.

Let

\[
\kappa_1 = (r - 1)^{s-2} (r - 2)^2 (s - 1)(s - 2)^2(s-2)
\]

and recall (5.7). Define the function \( \varphi : K \to \mathbb{R} \) by

\[
\varphi(\alpha, \beta, \gamma, \delta) = g(1 - \alpha - \beta) + 2(s - 1)g(\alpha) + \frac{s - 1}{s} g(rs - r - s - s\alpha) - g(\beta - \gamma) - 2g(\gamma) - 2g(\alpha - \beta - \delta) - g((s - 3)\alpha + \beta + \delta) - g(1 - \alpha - \beta - \gamma) + \alpha \ln(\kappa_1) + \beta \ln(\kappa_2) + \gamma \ln(\kappa_3) + \delta \ln(\kappa_4)
\]

where \( g(x) = x \ln x \) for \( x > 0 \), and \( g(0) = 0 \). We will need the following information about the function \( \varphi \). The proof of the following crucial result is lengthy and technical, so it is deferred to the appendix.

**Lemma 5.6.** Suppose that \( s \geq 3 \) and \( r > \rho(s) \), where \( \rho(s) \) is defined in Theorem 1.1. Then \( \varphi \) has a unique global maximum over the domain \( K \) which occurs at the point \( x^* = (\alpha^*, \beta^*, \gamma^*, \delta^*) \) defined by

\[
\alpha^* = \frac{rs - r - s}{r(s - 1)}, \quad \beta^* = \frac{rs - s - 2}{r(r - 1)(s - 1)}.
\]
\[
\gamma^* = \frac{2(r s - r - s)}{r(r - 1)^2(s - 1)^2}, \quad \delta^* = \frac{(r - 2)(r - 3)}{r(r - 1)(s - 1)}.
\]

The maximum value of \( \varphi \) equals
\[
\varphi(x^*) = \ln(r - 1) + \ln(s - 1) + \frac{(s - 1)(r s - r - s)}{s} \ln\left(\frac{r s - r - s}{r s - r}\right). \quad (5.13)
\]

Let \( Q(r, s) \) be as defined in Lemma 4.6, and denote by \( H^* \) the Hessian of \( \varphi(\alpha, \beta, \gamma, \delta) \) evaluated at the point \( x^* \). Then \( H^* \) is strictly negative definite and
\[
\det(-H^*) = \frac{r^5(r - 1)^5(r - 2)(s - 1)^8 Q(r, s)}{8(r - 3)(r s - r - s)^2(rs - 2r - 2s + 4)^2 h(r, s)}. \quad (5.14)
\]

With this result in hand, we may establish the following asymptotic expression for the second moment of \( Y \).

**Lemma 5.7.** Suppose that \( s \geq 3 \) and \( r > \rho(s) \) are fixed integers, where \( \rho(s) \) is defined in Theorem 1.1. Then as \( n \to \infty \) along \( I_{(r, s)} \),
\[
\frac{\mathbb{E}(Y^2)}{(\mathbb{E}Y)^2} \sim \frac{r(r s - r - s)}{(r - 2)\sqrt{Q(r, s)}}
\]
where \( Q(r, s) \) is defined in Lemma 4.6.

**Proof.** Firstly, extend the definition of \( J_t(a, b, c, d) \) to cover all \( (a, b, c, d) \in \mathcal{D} \), by defining \( \xi_t(a, b, c) = 1 \) if \( b = c = 0 \) and \( 1 \leq a \leq t - 1 \). We will apply Lemma 5.5 to calculate the sum of \( J_t(a, b, c, d) \) over the domain \( \mathcal{D} \). While doing so, we will observe that the contribution to the sum from \( \mathcal{D} \setminus \mathring{\mathcal{D}} \) is negligible, which will imply that the sum over the larger domain \( \mathcal{D} \) is also asymptotically equal to \( \mathbb{E}(Y^2)/(\mathbb{E}Y) \).

The first six conditions of Lemma 5.5 hold with the definitions given below.

(i) Let \( \mathcal{L} = \mathbb{Z}^4 \), a lattice with full rank \( m = 4 \) and with determinant 1.

(ii) The domain \( \mathcal{K} \) defined in (5.11) is compact, convex, is contained in \([0, 1]^4\) and has non-empty interior. Observe that \( \mathcal{D} = \mathbb{Z}^4 \cap tK \).

(iii) The function \( \varphi : \mathcal{K} \to \mathbb{R} \) defined before Lemma 5.6 is continuous. Furthermore, \( \varphi \) has a unique maximum at the point \( x^* \), by Lemma 5.6, and \( x^* \) belongs to the interior of \( \mathcal{K} \).

(iv) The function \( \varphi \) is infinitely differentiable in the interior of \( \mathcal{K} \). Let \( H^* \) denote the Hessian matrix of \( \varphi \) evaluated at \( x^* \). Then \( H^* \) is strictly negative definite, by Lemma 5.6.
(v) Write \( \varepsilon = (r(r - 1)^2(s - 1)^2)^{-1} \) and define

\[
K^* = \{(\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4 | \varepsilon < \gamma \leq \beta - \varepsilon, \ \varepsilon < \delta < \alpha - \beta - \varepsilon, \ \alpha + \beta + \gamma \leq 1 - \varepsilon\}.
\]

Then \( K^* \) is contained in the interior of \( K \) and \( x^* \in K^* \). Furthermore, the function \( \psi: K^* \to \mathbb{R} \) defined by

\[
\psi(\alpha, \beta, \gamma, \delta) = \frac{1}{(\alpha - \beta - \delta) \sqrt{\beta - \gamma} \delta} \frac{(1 - \alpha - \beta)(1 - \alpha - \beta - \gamma)((s - 3)\alpha + \beta + \delta)}{1 - \varepsilon}
\]

is continuous on \( K^* \). Direct substitution shows that

\[
\psi(x^*) = \frac{1}{2(s - 2)(rs - 2r - 2s + 4)} \sqrt{\frac{r^7 (r - 1)^5 (s - 1)^9}{2(r - 2)(r - 3) h(r, s)}} \tag{5.15}
\]

which is certainly positive when \( r > \rho(s) \) and \( s \geq 3 \).

(vi) Let \( \mathbf{w}_t \) be the zero vector for each \( t \).

It remains to prove that condition (vii) of Lemma 5.5 holds. Asymptotics are as \( t \to \infty \) along the set \( \{t \in \mathbb{Z}^+ : s \text{ divides } r(s - 1)t\} \). Define

\[
b_t = \frac{(s - 2)}{4\pi^2 t^2 \sqrt{s - 1}} \left( \frac{1}{(r - 1)(s - 1)} \frac{rs - r}{(rs - s)^2} \right)^{a t} \tag{5.16}
\]

and introduce the scaled variables

\[
\alpha = a/t, \quad \beta = b/t, \quad \gamma = c/t, \quad \delta = d/t. \tag{5.17}
\]

Observe that (5.8) holds when \( a = 0 \), since then \( \varphi(0, 0, 0, 0) = \frac{s - 1}{s} g(rs - r - s) \) and

\[
J_t(0, 0, 0, 0) = 1 \frac{1}{\mathbb{E}Y} = O(b_t) \exp \left( t \varphi(0, 0, 0, 0) + \frac{3}{2} \ln t \right),
\]

using Corollary 2.2. When \( a \geq 1 \), first rewrite all binomial coefficients in \( J_t(a, b, c, d) \) in terms of factorials (except those in the factor \( 1/\mathbb{E}Y \)), giving

\[
J_t(a, b, c, d) = \frac{\xi_t(a, b, c)}{2a^2} \left( (s(s - 1)(r - 1)^{s-2}(r - 2)^2)^a \kappa_2^a \kappa_3^c \kappa_4^d \right. \\
\times \frac{(a!)^2 (t - a - b)!((s - 2)a)!^2}{(b - c)! (c)!^2 ((a - b - d)!)^2 (t - a - b - c)! ((s - 3)a + b + d)!} \\
\times \left( (rn - s(t + a))! (rn/s - t)! 1 \\
\left( (rn/s - (t + a))! (rn - st)! \right) \mathbb{E}Y. \right.
\]

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Let \( x \vee y \) denote \( \max(x, y) \) and apply Stirling’s formula in the form

\[
\ln(N!) = N \ln N - N + \frac{1}{2} \ln(N \vee 1) + \frac{1}{2} \ln 2\pi + O(1/(N + 1)),
\]
valid for all integers \( N \geq 0 \), to the above expression for \( J_t(a, b, c, d) \). After substituting for \( \mathbb{E}Y \) using Corollary 2.2, this gives

\[
J_t(a, b, c, d)
\]

\[
= \left(1 + O \left( \frac{1}{c+1} + \frac{1}{d+1} + \frac{1}{b-c+1} + \frac{1}{a-b-d+1} + \frac{1}{t-a-b-c+1} \right) \right)
\times \frac{4\pi^2 (a-b-d) \sqrt{(s-1)(b-c) (t-a-b) (t-a-b-c)} ((s-3)a+b+d)}{(t-a-b)^{t-a-b} a^{2(s-1)a} (rs-r-s-sa/t)^{(s-1)(rs-r-s-sa/t)/s}}
\times \frac{1}{(r-1)(s-1)} \frac{1}{(rs-r-s)^2}^{(s-1)(rs-r-s)/s} \frac{t}{t}
\times \left( \frac{\psi(x^*)}{\sqrt{\det(-H^*)}} \right) b_t t^2 e^{t\varphi(x^*)},
\]

except that if some factor in the denominator is zero then it should be replaced by 1. (Also interpret \( 0^0 \) as 1.) Finally, rewriting this expression in terms of the scaled variables (5.17) proves that (5.8) and (5.9) hold. Hence condition (vii) of Lemma 5.5 is satisfied.

Therefore we may apply Lemma 5.5 to conclude that (5.10) holds: that is,

\[
\sum_{(a,b,c,d) \in \mathcal{D}} J_t(a, b, c, d) \sim \frac{4\pi^2 \psi(x^*)}{\sqrt{\det(-H^*)}} b_t t^2 e^{t\varphi(x^*)}, \tag{5.18}
\]

using the fact that the lattice \( Z^d \) has determinant 1. Observe that up to the \( 1 + o(1) \) relative error term, the answer depends only on \( \mathcal{D} \cap tK^* \), which equals \( \hat{\mathcal{D}} \cap tK^* \). (Indeed, the only terms which contribute non-negligibly to the sum are those which are close to \( x^* \).) Therefore we may replace \( \mathcal{D} \) by \( \hat{\mathcal{D}} \) in (5.18). By Lemma 5.4, the proof is completed by substituting (5.13) – (5.16) into (5.18).

\[\square\]

### 6 Proof of the threshold result

To prepare for the proof of Theorem 1.1, we now find the values of \( r, s \) for which \( \mathbb{E}Y \) tends to infinity. Define

\[
L(r, s) = \ln(r-1) + \ln(s-1) + \frac{(s-1)(rs-r-s)}{s} \ln \left( 1 - \frac{s}{rs-r} \right)
\]

and treat \( r \) as a continuous variable. Then \( L(r, s) \) is the natural logarithm of the base of the exponential factor in \( \mathbb{E}Y \), see Corollary 2.2. If \( L(r, s) \leq 0 \) then \( \mathbb{E}Y = o(1) \), so a.a.s. there are no loose Hamilton cycles. For example, \( L(3, 3) = 0 \), so a.a.s. \( F(n, 3, 3) \) has no loose Hamilton cycles. Similarly, if \( L(r, s) > 0 \) then \( \mathbb{E}Y \to \infty \).
Lemma 6.1. For any fixed integer \( s \geq 2 \), there exists a unique real number \( \rho(s) > 2 \) satisfying the lower and upper bounds given in (1.3) such that \( L(\rho(s), s) = 0 \),

\[
L(r, s) < 0 \quad \text{for} \quad r \in [2, \rho(s)) \quad \text{and} \quad L(r, s) > 0 \quad \text{for} \quad r \in (\rho(s), \infty).
\]

Proof. The statements hold when \( s \in \{2, 3\} \), as can be verified directly. For the remainder of the proof, assume that \( s \geq 4 \). Setting \( x = (rs - r - s)/s \), we may rewrite \( L(r, s) \) as \( L(r, s) = f_s(x) \) where

\[
f_s(x) = \ln(sx + 1) - (s - 1)x \ln \left( 1 + \frac{1}{x} \right).
\]

First we claim that \( f_s(x) \) is negative when \( x \in [1 - \frac{2}{s}, s - 1 - \frac{1}{s}] \). (This range of \( x \) corresponds to \( 2 \leq r \leq s + 1 \).) This can be verified directly for \( s = 4, \ldots, 13 \), while for fixed \( s \geq 14 \) we have, for \( x \) in this range,

\[
f_s(x) \leq \ln(sx + 1) - (s - 1)x \left( \frac{1}{x} - \frac{1}{2x^2} \right)
\leq \ln(s(s - 1)) - (s - 1) + \frac{s(s - 1)}{2(s - 2)} < 0.
\]

This implies that \( L(r, s) < 0 \) for all \( s \geq 4 \) and \( 2 \leq r \leq s + 1 \).

Next, suppose that \( x \geq s - 1 - \frac{1}{s} \). Then the derivative of \( f_s \) with respect to \( x \) satisfies

\[
f'_s(x) = \frac{s}{sx + 1} - (s - 1) \ln \left( 1 + \frac{1}{x} \right) + \frac{s - 1}{x + 1} \geq \frac{s}{sx + 1} - \frac{s - 1}{x} + \frac{s - 1}{x + 1}
= \frac{sx(x - s) + 2sx - (s - 1)}{(sx + 1)x(x + 1)} \geq \frac{sx(-1 - 1/s) + 2sx - (s - 1)}{(sx + 1)x(x + 1)} > 0.
\]

This implies that \( L(r, s) \) is monotonically increasing as a function of \( r \geq s + 1 \), for any fixed \( s \geq 4 \), as

\[
\frac{\partial}{\partial r} L(r, s) = \frac{(s - 1)}{s} f'_s(x).
\]

Furthermore, \( f_s(x) \) tends to infinity as \( x \to \infty \). Therefore the function \( L(\cdot, s) \) has precisely one root in \((2, \infty)\), for all \( s \geq 4 \). Let this root be \( r = \rho(s) \). Next we will prove that

\[
f_s(x_1) < 0 \quad \text{and} \quad f_s(x_2) > 0 \quad (6.1)
\]

where

\[
x_1 = \frac{e^{s-1}}{s} - \frac{s - 1}{2} - \frac{1}{s} - \frac{(s^2 - s + 1)^2}{se^{s-1}}, \quad x_2 = \frac{e^{s-1}}{s} - \frac{s - 1}{2} - \frac{1}{s}.
\]

Since \( \rho^-(s) = s(x_1 + 1)/(s - 1) \) and \( \rho^+(s) = s(x_2 + 1)/(s - 1) \), this will prove that \( \rho^-(s) < \rho(s) < \rho^+(s) \), as required.
When \( s = 4, 5 \), we can verify the inequalities (6.1) directly. Now suppose that \( s \geq 6 \).

Using the inequality \( a \ln(1 + 1/a) \geq 1 - \frac{1}{2a} \), which holds for all \( a > 1 \), we have

\[
\exp\left((s - 1)x \ln \left(1 + \frac{1}{x}\right)\right) \geq \exp\left(s - 1 - \frac{s - 1}{2x}\right)
\geq e^{s-1} \left(1 - \frac{s - 1}{2x}\right).
\]

Note that \( f_s(x) < 0 \) if this expression is bounded below by \( sx + 1 \). This holds if and only if

\[
2sx^2 - 2(e^{s-1} - 1)x + (s - 1)e^{s-1} < 0.
\]

Let \( x^- \) and \( x^+ \) denote the smaller and larger root of this quadratic (in \( x \)), respectively. Then

\[
x^+ = \frac{e^{s-1} - 1}{2s} + \frac{e^{s-1}}{2s} \sqrt{1 - \frac{2(e^s - s + 1)}{e^{s-1}} + \frac{1}{e^{2(s-1)}}}
\geq \frac{e^{s-1} - 1}{2s} + \frac{e^{s-1}}{2s} \sqrt{1 - \frac{2(e^s - s + 1)}{e^{s-1}}} \geq x_1
\]

using the inequality \((1 - a)^{1/2} \geq 1 - a/2 - a^2/2\), which holds for all \( a \in (0, 1) \). Also

\[
x^- < \frac{e^{s-1} - 1}{2s} < x_1,
\]

proving the first statement in (6.1).

For the upper bound, by definition of \( x_2 \), we have

\[
\ln(sx_2 + 1) = s - 1 + \ln \left(1 - \frac{s(s - 1)}{2e^{s-1}}\right) \geq (s - 1) \left(1 - \frac{s}{2e^{s-1}} - \frac{s^2(s - 1)}{6e^{2(s-1)}}\right)
\]

since \( \frac{s(s-1)}{2e^{s-1}} < 1/3 \) when \( s \geq 6 \), and \( \ln(1 - a) \geq -a - 2a^2/3 \) when \( 0 < a < 1/3 \). Next, since \( x_2 > 1 \) we have

\[
(s - 1)x_2 \ln \left(1 + \frac{1}{x_2}\right) \leq (s - 1) \left(1 - \frac{1}{2x_2} + \frac{1}{3x_2^2}\right).
\]

Hence \( f(x_2) > 0 \) holds if

\[
\frac{1}{3x_2^2} \leq \frac{1}{2x_2} - \frac{s}{2e^{s-1}} - \frac{s^2(s - 1)}{6e^{2(s-1)}}.
\] (6.2)

Substituting the expression for \( x_2 \), the left hand side of (6.2) becomes

\[
\frac{4s^2}{3(2e^{s-1} - (s^2 - s + 2))^2}
\]
while the right hand side of (6.2) becomes

\[
\frac{s(s^2 - s + 2)}{2e^{s-1}(2e^{s-1} - (s^2 - s + 2))} - \frac{s^2(s - 1)}{6e^{2(s-1)}} > \frac{s(s^2 - s + 2)}{4e^{2(s-1)}} - \frac{s^2(s - 1)}{6e^{2(s-1)}}
\]

\[
= \frac{s(s^2 - s + 2)}{12e^{2(s-1)}}.
\]

Therefore, it suffices to prove that

\[
16s^2 e^{2(s-1)} \leq s(s^2 - s + 6) (2e^{s-1} - (s^2 - s + 2))^2.
\]

But the right hand side of this expression is bounded below by \(4 \cdot \frac{49}{64} s(s^2 - s + 6) e^{2(s-1)}\) when \(s \geq 6\). For \(s \geq 6\) the inequality \(256s \leq 49(s^2 - s + 6)\) holds, and hence (6.2) holds. Therefore \(f_s(x_2) > 0\) for \(s \geq 6\), completing the proof.

We may now complete the proof of our main result, Theorem 1.1, establishing a threshold result for existence of a loose Hamilton cycle in \(G(n, r, s)\).

**Proof of Theorem 1.1.** When \(s = 2\), the result follows immediately from Robinson and Wormald [12, 13], since \(\rho(2) < 3\). Now suppose that \(s \geq 3\). Lemma 6.1 proves that there is a unique value of \(\rho(s) \geq 3\) such that

\[
(r - 1)(s - 1) \left(\frac{rs - r - s}{rs - r}\right)^{(s-1)(rs-r-s)/s} = 1.
\]

Furthermore, Lemma 6.1 proved that the upper and lower bounds on \(\rho(s)\) given in (1.3) hold.

If \(r \geq 3\) is an integer with \(r \leq \rho(s)\) then a.a.s. \(G(n, r, s)\) contains no loose Hamilton cycle, using (2.5) and Lemma 2.2, since \(\Pr(Y > 0) \leq \mathbb{E}Y\). (This can also be deduced from Theorem 1.2.)

Now suppose that \(r > \rho(s)\) for some fixed \(s \geq 3\). As noted in Section 1, Cooper et al. [2] proved that condition (A1) of Theorem 2.4 holds, with \(\lambda_k\) as defined in (2.8). The remaining conditions of Theorem 2.4 have also been established: Corollary 4.4 proves that condition (A2) holds, Lemma 4.6 shows that condition (A3) holds, while condition (A4) is shown to hold by combining Corollary 4.4 and Lemma 5.7. Also recall that \(\delta_k > -1\) for all \(k \geq 1\), by Lemma 4.5. Therefore Lemma 2.6 implies that a.a.s. \(Y_G > 0\). This shows that a.a.s. \(G \in G(n, r, s)\) contains a loose Hamilton cycle whenever \(s \geq 3\) and \(r > \rho(s)\), completing the proof.

Finally, we provide the asymptotic distribution of the number of loose Hamilton cycles in \(G(n, r, s)\).
Theorem 6.2. Suppose that $r$, $s$ are integers with $s \geq 2$ and $r > \rho(s)$. Recall that $Y_G$ is the number of loose Hamilton cycles in $G(n,r,s)$. Then with $\zeta_1, \zeta_2 \in \mathbb{C}$ satisfying (4.3),

$$
\frac{Y_G}{\mathbb{E}Y_G} \xrightarrow{d} \prod_{k=2}^{\infty} \left(1 + \frac{\zeta_1^k + \zeta_2^k - 1}{((r-1)(s-1))^k}\right)^{Z_k} \exp\left(\frac{1 - \zeta_1^k - \zeta_2^k}{2k}\right)
$$

where the variables $Z_k$ are independent Poisson variables with

$$
\mathbb{E}Z_k = \frac{((r-1)(s-1))^k}{2k}
$$

for $k \geq 2$.

Proof. The case $s = 2$ was proved by Janson [11, Theorem 2]. (To see that our expression matches his, observe that when $s = 2$ we have $\{\zeta_1, \zeta_2\} = \{0, -1\}$. Hence the factor of $W$ corresponding to any even value of $k$ equals 1.) For $s \geq 3$, we showed in the proof of Theorem 1.1 that conditions (A1)–(A4) of Theorem 2.4 hold for $Y$. Then the result follows by Lemma 2.6.

For possible future reference we remark that our arguments prove, as an intermediate step, that the analogues of Theorem 1.1 and Theorem 6.2 for $Y$ also hold, with the same threshold function $\rho(s)$. (In the analogue of Theorem 6.2 for $Y$, the product begins at $k = 1$, not $k = 2$.)

References


A Search for the global maximum

We now present the proof of Lemma 5.6.

Observe that $K$, defined in (5.11), is a compact, convex set in $[0, 1]^4$. Furthermore, $\varphi$, defined in (5.12), is a continuous function on $K$. Therefore $\varphi$ attains its maximum value at least once. Moreover, $\varphi$ is infinitely differentiable in the interior of $K$. The first-order partial derivatives of $\varphi$ are given by

$$\frac{\partial \varphi}{\partial \alpha} = -\ln(1 - \alpha - \beta) + 2(s-1)\ln(\alpha) - 2\ln(\alpha - \beta - \delta) + \ln(1 - \alpha - \beta - \gamma) - (s-3)\ln((s-3)\alpha + \beta + \delta) - (s-1)\ln(rs - r - s - s\alpha) + \ln(\kappa_1), \quad (A.1)$$

$$\frac{\partial \varphi}{\partial \beta} = -\ln(1 - \alpha - \beta) + 2\ln(\alpha - \beta - \delta) - \ln(\beta - \gamma) + \ln(1 - \alpha - \beta - \gamma) - \ln((s-3)\alpha + \beta + \delta) + \ln(\kappa_2), \quad (A.2)$$

$$\frac{\partial \varphi}{\partial \gamma} = \ln(\beta - \gamma) - 2\ln(\gamma) + \ln(1 - \alpha - \beta - \gamma) + \ln(\kappa_3), \quad (A.3)$$

$$\frac{\partial \varphi}{\partial \delta} = -\ln(\delta) + 2\ln(\alpha - \beta - \delta) - \ln((s-3)\alpha + \beta + \delta) + \ln(\kappa_4). \quad (A.4)$$
For $\tau \geq 1$, let
\[
p_{\tau} = \frac{(\tau - 1)^2 + \kappa_3(\tau - 1)}{(\tau - 1)^2 + 2\kappa_3\tau - \kappa_3}, \quad q_{\tau} = \frac{\kappa_4(\tau - 1)^2}{\kappa_2((\tau - 1)^2 + 2\kappa_3\tau - \kappa_3)}
\]
and define $x_{\tau} = (\alpha_{\tau}, \beta_{\tau}, \gamma_{\tau}, \delta_{\tau})$ by
\[
\alpha_{\tau} = \frac{p_{\tau} + q_{\tau} + \frac{(s-3)}{2\kappa_4} q_{\tau} + \sqrt{\frac{(s-2)}{\kappa_4} q_{\tau}(p_{\tau} + q_{\tau}) + \frac{(s-3)^2}{4\kappa_4^2} q_{\tau}^2}}{1 + p_{\tau} + q_{\tau} + \frac{(s-3)}{2\kappa_4} q_{\tau} + \sqrt{\frac{(s-2)}{\kappa_4} q_{\tau}(p_{\tau} + q_{\tau}) + \frac{(s-3)^2}{4\kappa_4^2} q_{\tau}^2}}\]
\[
\beta_{\tau} = p_{\tau}(1 - \alpha_{\tau}), \quad \gamma_{\tau} = \frac{\kappa_3(\tau-1)}{(\tau-1)^2 + 2\kappa_3\tau - \kappa_3}(1 - \alpha_{\tau}), \quad \delta_{\tau} = q_{\tau}(1 - \alpha_{\tau}).
\]
Since $\kappa_2, \kappa_3, \kappa_4 > 0$ it follows immediately that $\alpha_1 = 0$ and $\lim_{\tau \to \infty} \alpha_{\tau} = 1$. Next, observe that both $p_{\tau}$ and $q_{\tau}/p_{\tau}$ are increasing, infinitely differentiable functions of $\tau \in [1, \infty)$, and hence
\[
\alpha_{\tau} \text{ is a strictly increasing differentiable function of } \tau \in [1, \infty). \tag{A.5}
\]
Observe also that
\[
x_{\tau} \text{ lies in the interior of the domain } K, \text{ for any } \tau > 1. \tag{A.6}
\]
Recalling that $\kappa_4 < 1$, we have
\[
\sqrt{\frac{(s-2)}{\kappa_4} q_{\tau}(p_{\tau} + q_{\tau}) + \frac{(s-3)^2}{4\kappa_4^2} q_{\tau}^2} \geq q_{\tau} + \frac{(s-3)}{2\kappa_4} q_{\tau}, \tag{A.7}
\]
and hence
\[
(1 - \alpha_{\tau})^{-1} \geq 1 + p_{\tau} + \left(2 + \frac{s-3}{\kappa_4}\right) q_{\tau} \geq 1 + p_{\tau} + (s-1)q_{\tau}. \tag{A.8}
\]
We also note that $\tau = \frac{1 - \alpha_{\tau} - \beta_{\tau}}{1 - \alpha_{\tau} - \beta_{\tau} - \gamma_{\tau}}$ for all $\tau \geq 1$.

Our interest in this particular curve $x_{\tau}$ is clarified by the following lemma, which shows that $x_{\tau}$ is a parameterisation of a ridge which must contain any global maximum of $\varphi$. We prove Lemma A.1 in Section A.1.

**Lemma A.1.** Suppose that the assumptions of Lemma 5.6 hold. For $\tau \geq 1$, the function $\eta_{\tau}(\beta, \gamma, \delta) = \varphi(\alpha_{\tau}, \beta, \gamma, \delta)$ has a unique global maximum over the domain
\[
K_{\tau} = \{(\beta, \gamma, \delta) \in \mathbb{R}^3 \mid 0 \leq \gamma \leq \beta, \; 0 \leq \delta \leq \alpha_{\tau} - \beta, \; \beta + \gamma \leq 1 - \alpha_{\tau}\}
\]
which occurs at the point $(\beta_{\tau}, \gamma_{\tau}, \delta_{\tau})$. In particular, $\frac{\partial}{\partial \beta} \varphi(x_{\tau}) = \frac{\partial}{\partial \gamma} \varphi(x_{\tau}) = \frac{\partial}{\partial \delta} \varphi(x_{\tau}) = 0$ for any $\tau > 1$.  

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Recall the point $x^*$ from the statement of Lemma 5.6. Note that $x^*$ belongs to the ridge, namely, $x^* = x_{r^*}$ for $r^* = (r - 1)(s - 1)$. Therefore, by (A.6), the point $x^*$ belongs to the interior of $K$. Direct substitution shows that (5.13) and (5.14) hold, and combining the latter with Lemma 4.6 shows that $det(-H^*) > 0$. Direct substitution also shows that $\frac{\partial}{\partial \alpha} \varphi(x^*) = 0$, so $x^*$ is a stationary point of $\varphi$. Lemma A.1 implies that all eigenvalues of $-H^*$ are strictly positive, since otherwise there should be at least two negative eigenvalues and then $(\beta^*, \gamma^*, \delta^*)$ could not maximize $\eta_{r^*}$. In particular, this shows that $-H^*$ is positive definite, and hence $H^*$ is negative definite, as claimed. Therefore, $x^*$ is positive definite. It remains to prove that $\varphi(x^*)$ is strictly larger than $\varphi(x_{r^*})$ for all $\tau \geq 1$ with $\tau \neq \tau^*$.

First we consider $\tau = 1$. Note that $\alpha_1 = \beta_1 = \gamma_1 = \delta_1 = 0$, therefore

$$\varphi(x_1) = \frac{(s - 1)(rs - r - s)}{s} \ln(rs - r - s).$$

Using (5.13), we find that

$$\varphi(x^*) - \varphi(x_1) = \ln(r - 1) + \ln(s - 1) + \frac{(s - 1)(rs - r - s)}{s} \ln \left(\frac{rs - r - s}{rs - r}\right), \quad (A.9)$$

and this expression is positive since $r > \rho(s)$ (see Lemma 6.1).

Now we may suppose that $\tau > 1$. Let $\varphi_\alpha$ denote the partial derivative of $\varphi$ with respect to $\alpha$. By Lemma A.1 we have $\frac{\partial}{\partial \alpha} \varphi(x_r) = 0$. Combining (A.1) and (A.4), we find that

$$\varphi_\alpha(x_r) = -\ln(1 - \alpha_\tau - \beta_\tau) + 2(s - 1) \ln \alpha_\tau - 2 \ln(\alpha_\tau - \beta_\tau - \delta_\tau) + \ln(1 - \alpha_\tau - \beta_\tau - \gamma_\tau) - (s - 3) \ln((s - 3)\alpha_\tau + \beta_\tau + \delta_\tau) - (s - 1) \ln(rs - r - s - s\alpha_\tau) + \ln \kappa_1$$

$$= -\ln \tau - (s - 3) \ln((s - 3)\alpha_\tau + \beta_\tau + \delta_\tau) - \ln \left(\frac{\delta_\tau((s - 3)\alpha_\tau + \beta_\tau + \delta_\tau)}{\alpha_\tau^2}\right)$$

$$+ 2(s - 2) \ln \alpha_\tau - (s - 1) \ln(rs - r - s - s\alpha_\tau) + \ln \kappa_4 + \ln \kappa_1.$$ 

We will use primes to denote differentiation with respect to $\tau$.

**Lemma A.2.** Suppose that the assumptions of Lemma 5.6 hold. The functions

$$\alpha_\tau + \beta_\tau + \delta_\tau, \quad \frac{\delta_\tau((s - 3)\alpha_\tau + \beta_\tau + \delta_\tau)}{\alpha_\tau^2}$$

are both strictly increasing with respect to $\tau \in [1, \infty)$.

**Proof.** Recalling that $p_\tau$ and $q_\tau$ both increase with $\tau$, it follows that

$$\frac{\delta_\tau}{\alpha_\tau} = \left(1 + p_\tau/q_\tau + \frac{(s - 3)}{2\kappa_4} + \sqrt{\frac{(s - 2)(1 + p_\tau/q_\tau)}{\kappa_4} + \frac{(s - 3)^2}{4\kappa_4^2}}\right)^{-1}$$

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and
\[
\frac{\delta_r((s-3)\alpha_r + \beta_r + \delta_r)}{\alpha_r^2} = \kappa_4 \left( 1 - \frac{1 - (s-3)}{1 + \frac{(s-3)}{2\kappa_4(1+p_r/q_r)} + \frac{(s-2)}{\kappa_4(1+p_r/q_r)} + \frac{(s-3)^2}{4\kappa_4^2(1+p_r/q_r)^2}} \right)^2
\]
are both increasing functions of \( \tau \). Hence, by (A.5), it follows that \( \delta_r \) increases. Finally, observe that \( p_r < 1 \) for all \( \tau \geq 1 \), so
\[
\frac{d}{d\tau} (\alpha_r + \beta_r + \delta_r) = \alpha_r' - \alpha_r'p_r + (1 - \alpha_r)p_r' + \delta_r' > 0,
\]
completing the proof. \( \square \)

Using Lemma A.2 and the fact that \( \beta_r + \delta_r \leq \alpha_r \leq 1 \), we calculate that
\[
\varphi'(x_{\tau}) \leq -\frac{1}{\tau} - (s-3)(s-3)\alpha_r' + \beta_r' + \delta_r' + 2(s-2)\alpha_r' \frac{\alpha_r}{\alpha_r} + \frac{(s-1)s\alpha_r'}{rs-r-s-s\alpha_r}
\]
\[
\leq -\frac{1}{\tau} + \left( 2(s-2) - \frac{(s-3)(s-4)}{s-2} + \frac{(s-1)s}{rs-r-2s} \right) \alpha_r'. \tag{A.10}
\]
Observe that, by Lemma A.1,
\[
\varphi'(x_{\tau}) = \varphi_{\alpha}(x_{\tau}) \alpha_r'. \tag{A.11}
\]
Using (A.5), it follows that \( \varphi'(x_{\tau}) \) and \( \varphi_{\alpha}(x_{\tau}) \) always have the same sign. Hence, the next lemma implies that the function \( \tau \mapsto \varphi(x_{\tau}) \) is concave when \( \tau \) is large enough. We prove Lemma A.3 in Section A.2 below.

**Lemma A.3.** Suppose that \( s \geq 3 \) and \( r > \rho(s) \). Then \( \varphi_{\alpha}'(x_{\tau}) < 0 \) whenever one of the following conditions holds:

(i) \( \tau \geq 2(s+2)^2 \),

(ii) \( (r - 1)(s - 1) \geq (s + 1)^3 \) and \( \tau \geq (s + 1)^{2.5} \).

The rest of the argument is split into two cases.

**Case 1:** \( r \) is sufficiently large

Assume that \( (r - 1)(s - 1) \geq (s + 1)^3 \). Since \( \varphi_{\alpha}(x^*) = 0 \), it follows from Lemma A.3(ii) that
\[
\varphi_{\alpha}(x_{\tau}) > 0 \text{ for } (s + 1)^{2.5} \leq \tau < \tau^* \quad \text{and} \quad \varphi_{\alpha}(x_{\tau}) < 0 \text{ for } \tau > \tau^*. \tag{A.12}
\]
By (A.11), this shows that \( \varphi(x_\tau) \) is a concave function of \( \tau \) when \( \tau \geq (s + 1)^3 \). So there can be at most one local maximum of \( \varphi(x_\tau) \) in \([ (s + 1)^3, \infty) \), and we know that this local maximum occurs at \( \tau = \tau^* \). Combining this with Lemma A.1 and (A.9), we conclude in particular that there are no global maxima of \( \varphi \) on the boundary of \( K \).

To complete the proof in this case, we will show that \( \varphi(x_\tau) < \varphi(x^*) \) for any \( x_\tau \neq x^* \) such that \( \varphi_\alpha(x_\tau) = 0 \) and \( \tau \neq \tau^* \). That is, we may restrict our attention to values of \( \tau \) which correspond to stationary points of \( \varphi \). For a contradiction, suppose that \( \varphi(x_{\tilde{\tau}}) \geq \varphi(x^*) \) for some \( \tilde{\tau} > 1 \) with \( \tilde{\tau} \neq \tau^* \) and \( \varphi_\alpha(x_{\tilde{\tau}}) = 0 \). Then \( 1 < \tilde{\tau} < (s + 1)^{2.5} \), by (A.12). We solve the equations \( \frac{\partial \varphi}{\partial \alpha}(x_{\tilde{\tau}}) = \frac{\partial \varphi}{\partial \gamma}(x_{\tilde{\tau}}) = \frac{\partial \varphi}{\partial \delta}(x_{\tilde{\tau}}) = 0 \) for \( \ln \kappa_1 \), \( \ln \kappa_2 \), \( \ln \kappa_3 \), \( \ln \kappa_4 \), and then substitute these expressions into (5.12), to obtain

\[
\varphi(x_{\tilde{\tau}}) = \ln \tilde{\tau} + \frac{s - 1}{s} (rs - r - s) \ln (rs - r - s - s\tilde{\alpha})
\]

where \( \tilde{\alpha} = \alpha_{\tilde{\tau}} \). By the lower bound on \( r \) we have

\[
r \geq \frac{(s + 1)^3}{s - 1} + 1 \geq 33 \quad \text{and} \quad \frac{rs - r - s}{rs - r - 2s} \leq 1 + \frac{s}{(s + 1)^3 - s - 1} \leq 1.05. \tag{A.13}
\]

Then

\[
0 \leq \varphi(x_{\tilde{\tau}}) - \varphi(x_1) = \ln \tilde{\tau} + \frac{s - 1}{s} (rs - r - s) \ln \left(1 - \frac{s\tilde{\alpha}}{rs - r - s}\right)
= \ln \tilde{\tau} - (s - 1)^2 \tilde{\alpha} \tag{A.14}
\]

and

\[
0 \leq \varphi(x_{\tilde{\tau}}) - \varphi(x^*) = \ln \tilde{\tau} - \ln \tau^* + \frac{s - 1}{s} (rs - r - s) \ln \left(\frac{rs - r - s - s\tilde{\alpha}}{rs - r - s - s\alpha^*}\right)
\leq \ln \tilde{\tau} - 3 \ln (s + 1) + \frac{s - 1}{s} (rs - r - s) \ln \left(1 + \frac{s(1 - \tilde{\alpha})}{rs - r - 2s}\right)
\leq \ln \tilde{\tau} - 3 \ln (s + 1) + (s - 1) \frac{rs - r - s}{rs - r - 2s} (1 - \tilde{\alpha})
\leq \ln \tilde{\tau} - 3 \ln (s + 1) + 1.05 (s - 1)(1 - \tilde{\alpha}), \tag{A.15}
\]

using (A.13) for the final inequality. Taking a carefully chosen linear combination of the inequalities (A.14), (A.15), we conclude that

\[
0 \leq 1.05(1 - \tilde{\alpha})(\varphi(x_{\tilde{\tau}}) - \varphi(x_1)) + \tilde{\alpha}(\varphi(x_{\tilde{\tau}}) - \varphi(x_1))
\leq 1.05 \ln \tilde{\tau} - 3\tilde{\alpha} \ln (s + 1). \tag{A.16}
\]

Since \( \tilde{\tau} < (s + 1)^{2.5} \), this implies that

\[
\tilde{\alpha} \leq \frac{1.05 \ln(\tilde{\tau})}{3 \ln(s + 1)} < \frac{1.05 \times 2.5}{3} = 0.875. \tag{A.17}
\]
Now observe that by (A.8),

\[ 1 - (1 - \tilde{\alpha})^{-1} + p\tilde{\tau} + q\tilde{\tau} \leq 0. \]

In Lemma A.4, stated later, we will prove that

\[ 1 - (1 - \tilde{\alpha})^{-1} + p\tilde{\tau} + q\tilde{\tau} \geq T(\tilde{\alpha}) \quad (A.18) \]

where

\[ T(\alpha) = -(1 - \alpha)^{-1} + 1 + \frac{(50^{\alpha} - 1)^2 + 2(50^{\alpha} - 1)}{(50^{\alpha} - 1)^2 + 4(50^{\alpha} - 1) + 2} + R(\alpha) \]

and

\[ R(\alpha) = \begin{cases} 
0 & \text{if } \alpha < 0.35, \\
\frac{15}{62} 50^{\alpha}(50^{\alpha} - 1)^2 / (50^{\alpha} - 1)^2 + 4(50^{\alpha} - 1) + 2 & \text{otherwise.}
\end{cases} \]

From the plot of the function \( \alpha \mapsto T(\alpha) \) given in Figure 3, we observe that \( T(\alpha) \) is strictly positive for \( 0 < \alpha \leq 0.875 \). Therefore, using (A.17), if (A.18) holds then

\[ 1 - (1 - \tilde{\alpha})^{-1} + p\tilde{\tau} + q\tilde{\tau} > 0. \]

But this contradicts (A.8). Therefore no other local maximum \( \tilde{\tau} \) of \( \varphi(x_\tau) \) can exist in the interval \([1, (s + 1)^{2.5}]\).

To complete the proof of the lemma in the case that \((r - 1)(s - 1) \geq (s + 1)^3\), we must establish (A.18).

**Lemma A.4.** If \( s \geq 3 \) and \((r - 1)(s - 1) \geq (s + 1)^3\) then (A.18) holds.
Proof. Recalling the definition of $\kappa_2$ in (5.7), and since
\[ r - 2 \geq \frac{(s + 1)^3}{s - 1} - 1 \geq (s + 2)^2, \]
it follows that
\[
\kappa_2 = s^2 - s + \frac{2(s-2)(s-1)}{r-2} + \frac{(s-2)(s-3)}{(r-2)^2}
\leq s^2 - s + \frac{2(s-1)(s-2)}{(s+2)^2} + \frac{(s-2)(s-3)}{(s+2)^4}
\leq s^2 - 1.
\]
(A.19)
(A.20)

By (A.16), since $s \geq 3$ we have
\[ \tilde{\tau} \geq (s + 1)^{\frac{3}{10}} \tilde{\alpha} \geq 50^{\tilde{\alpha}}, \]
while if $\tilde{\alpha} \geq 0.35$ then we can instead write
\[ \tilde{\tau} \geq (s + 1)4^{\frac{3}{10}}\tilde{\alpha}^{-1} \geq \frac{s+1}{4} 50^{\tilde{\alpha}}. \]

Noting that $\kappa_3 \leq 2$, we can estimate
\[ p_{\tilde{\tau}} \geq \frac{(50^{\tilde{\alpha}} - 1)^2 + 2(50^{\tilde{\alpha}} - 1)}{(50^{\tilde{\alpha}} - 1)^2 + 4(50^{\tilde{\alpha}} - 1) + 2} \]
and if $\tilde{\alpha} \geq 0.35$ we have
\[ q_{\tilde{\tau}} \geq \frac{\kappa_4(s + 1)}{4\kappa_2} \frac{50^{\tilde{\alpha}}}{(50^{\tilde{\alpha}} - 1)^2 + 4(50^{\tilde{\alpha}} - 1) + 2}. \]

Additionally, it follows from (A.13) that $\kappa_4 = \frac{r-3}{r-2} \geq \frac{30}{31}$. Using these inequalities, together with the upper bound on $\kappa_2$ given in (A.20), we conclude that (A.18) holds, as required.

Case 2: $r$ is small

It remains to consider the case that $(r - 1)(s - 1) < (s + 1)^3$. By definition of $\rho(s)$ (see Table 1), the only remaining pairs $(r, s)$ belong to the set
\[ \mathcal{A} = \{(r, 3) \mid r = 4, \ldots, 32\} \cup \{(r, 4) \mid r = 6, \ldots, 41\} \cup \{(r, 5) \mid r = 12, \ldots, 54\} \]
\[ \cup \{(r, 6) \mid r = 28, \ldots, 69\} \cup \{(r, 7) \mid r = 65, \ldots, 86\}. \]

There are 172 pairs $(r, s)$ in $\mathcal{A}$. (It may be possible to reduce this number by refining the above analysis, but we have not pursued this.)

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For $(r, s) \in A$ we consider two functions. The first is
\[ \tau \mapsto \tau^{-(s-4)/(s-2)} (\varphi(x^*) - \varphi(x_\tau)) \]
on the interval $\tau \in [1, s + 1]$, and the second is
\[ \tau \mapsto -\frac{\tau}{\ln(\tau)} \varphi'_a(x_\tau) \]
on the interval $\tau \in [s + 1, 2(s + 2)^2]$. Figures 4–8 show the plots of these functions for all $(r, s) \in A$, with all pairs with a given value of $s$ displayed together.

![Figures 4–8](image)

Figure 4: The two plots for $s = 3, r = 4, \ldots, 32$.

In each of these figures, the plot of the first function is shown on the left, and the plot of the second function is shown on the right. The colours are used to help distinguish between the plots for different values of $r$. (The scaling factors $\tau^{-(s-4)/(s-2)}$ and $\tau/\ln(\tau)$ in the first and second plot, respectively, do not affect the sign of the functions, and are included to attempt to spread out the different plots shown in each figure.)

Consider each $(r, s) \in A$ in turn: we can see that both plots are strictly positive over the given intervals. The first plot (on the left) shows that $\varphi(x_\tau)$ is strictly less than $\varphi(x^*)$ for all $\tau \in [1, s + 1]$.

Combining the second plot (on the right) with Lemma A.3(i), using (A.11), we conclude that the function $\tau \mapsto \varphi(x_\tau)$ is concave for $\tau \geq s + 1$. Therefore $\varphi(x_\tau)$ has at most one local maximum in the interval $[s + 1, \infty)$. 
Figure 5: The two plots for $s = 4, r = 6, \ldots, 41$.

Figure 6: The two plots for $s = 5, r = 12, \ldots, 54$.

But we know that $x^* = x_{\tau^*}$ is a local maximum of $\varphi(x_{\tau})$, and $\tau^* = (r - 1)(s - 1) > s + 1$ when $s \geq 3$ and $r \geq s + 1$. Combining all this, we conclude that $x^*$ is the unique global maximum of $\varphi(x_{\tau})$ on $[1, \infty)$. 
This argument covers the 172 remaining cases of \((r,s) \in A\) and completes the proof of Lemma 5.6.
A.1 Proof of Lemma A.1

Since \( \eta_r \) is continuous and \( K_r \) is compact, the function \( \eta_r \) attains its maximum at least once on \( K_r \). For \( \tau = 1 \) the region \( K_r \) consist of one point \((\beta_r, \gamma_r, \delta_r)\), so the lemma is true for this case. In the following we assume that \( \tau > 1 \), which implies that \( 0 < \alpha_r < 1 \).

Let

\[
G_{\tau, \beta}(\gamma) = \frac{\gamma^2}{(\beta - \gamma)(1 - \alpha_r - \beta - \gamma)}, \quad D_{\tau, \beta}(\delta) = \frac{((s - 3)\alpha_r + \beta + \delta)\delta}{(\alpha_r - \beta - \delta)^2}.
\]

Note that \( G_{\tau, \beta}(\gamma) = \kappa_3 \) if and only if \( \frac{\partial G}{\partial \gamma}(\alpha_r, \beta, \gamma, \delta) = 0 \), by (A.3). For any \( 0 < \beta < \min\{\alpha_r, 1 - \alpha_r\} \), the function \( G_{\tau, \beta}(\gamma) \) is strictly increasing with respect to \( \gamma \) and

\[
G_{\tau, \beta}(0) = 0, \quad \lim_{\gamma \to \min\{\beta, 1 - \alpha_r - \beta\}} G_{\tau, \beta}(\gamma) = \infty.
\]

Therefore, there is a unique value of \( \gamma \), say \( \gamma = \hat{\gamma}_r(\beta) \), which satisfies \( G_{\tau, \beta}(\gamma) = \kappa_3 \). Note that \( \hat{\gamma}_r(\beta) < \beta \) for all \( \beta \in (0, \min\{\alpha_r, 1 - \alpha_r\}) \), since \( \kappa_3 \) is finite. To cover cases \( \beta = 0 \) and \( \beta = 1 - \alpha_r \), we continuously extend \( \hat{\gamma}_r \) and put \( \hat{\gamma}_r(0) = \hat{\gamma}_r(1 - \alpha_r) = 0 \).

Similarly \( D_{\tau, \beta}(\delta) = \kappa_4 \) if and only if \( \frac{\partial D}{\partial \delta}(\alpha_r, \beta, \gamma, \delta) = 0 \), by (A.4). For any \( 0 \leq \beta < \min\{\alpha_r, 1 - \alpha_r\} \), the function \( D_{\tau, \beta}(\delta) \) is strictly increasing with respect to \( \delta \) and

\[
D_{\tau, \beta}(0) = 0, \quad \lim_{\delta \to \alpha_r - \beta} D_{\tau, \beta}(\delta) = \infty.
\]

Therefore, there is a unique value of \( \delta \), say \( \delta = \hat{\delta}_r(\beta) \), such that \( D_{\tau, \beta}(\delta) = \kappa_4 \). Note that \( \hat{\delta}_r(\beta) < \alpha_r - \beta \) for all \( \beta \in [0, \min\{\alpha_r, 1 - \alpha_r\}] \), since \( \kappa_4 \) is finite. We continuously extend \( \hat{\delta}_r \) by defining \( \hat{\delta}_r(\alpha_r) = 0 \).

We now show that no local maximum of \( \eta_r \) can lie on a boundary of \( K_r \). Suppose that \((\beta_x, \gamma_x, \delta_x)\) is a local maximum of \( \eta_r \) in \( K_r \).

(i) If \( \beta_x = 0 \) then \( \gamma_x = 0 \) and \( \delta_x = \hat{\delta}_r(0) \in (0, \alpha_r) \). (The fact that \( \hat{\delta}_r(0) < \alpha_r \) arises by definition of \( \hat{\delta}_r(0) \), since \( \kappa_4 \) is finite.) From (A.2), we find that, for \( \beta \) in the neighbourhood of 0,

\[
\frac{\partial \eta_r}{\partial \beta}(\beta, \gamma_x, \delta_x) = - \ln \beta + O(1) > 0.
\]

Therefore, the point \((\beta_x, \gamma_x, \delta_x)\) cannot be a local maximum of \( \eta_r(\beta, \gamma_x, \delta_x) \) if \( \beta_x = 0 \).

(ii) If \( \beta_x = \alpha_r < 1 - \alpha_r \) then \( \delta_x = 0 \) and \( \gamma_x = \hat{\gamma}_r(\alpha_r) \in (0, \alpha_r) \). From (A.2), we find that, for \( \beta \) in the neighbourhood of \( \alpha_r \),

\[
\frac{\partial \eta_r}{\partial \beta}(\beta, \gamma_x, \delta_x) = 2 \ln(\alpha_r - \beta) + O(1) < 0.
\]

Therefore, the point \((\beta_x, \gamma_x, \delta_x)\) cannot be a local maximum of \( \eta_r(\beta, \gamma_x, \delta_x) \) if \( \beta_x = \alpha_r \) and \( 0 < \alpha_r < 1/2 \).
(iii) If \( \beta_x = 1 - \alpha_x \leq \alpha_x \) then \( \gamma_x = 0 \) and \( \delta_x = \tilde{\delta}_x(1 - \alpha_x) \). From (A.2), (A.3) and \( \frac{\partial \eta}{\partial q} = 0 \), we find that for \( \beta \) in the neighbourhood of \( 1 - \alpha_x \),

\[
\frac{\partial \eta}{\partial \beta}(\beta, \tilde{\gamma}_x(\beta), \delta_x) = -\ln\left(\frac{1 - \alpha_x - \beta}{1 - \alpha_x - \beta + \tilde{\gamma}_x(\beta)}\right) + 2\ln(\alpha_x - \beta - \delta_x) + O(1)
\]

\[
\leq -\ln\left(1 + \frac{\beta - \tilde{\gamma}_x(\beta)}{\kappa_3 \tilde{\gamma}_x(\beta)}\right) + O(1)
\]

\[
= -\ln\left(1 + \frac{\kappa_3(\beta - \tilde{\gamma}_x(\beta))}{\gamma_x (\beta - \tilde{\gamma}_x(\beta))}\right) + O(1) < 0.
\]

(We only need to take care with the term \( 2\ln(\alpha_x - \beta - \delta_x) \) when \( \alpha_x = 1/2 \), in which case \( \delta_x = 0 \). But since this term is negative we may bound it above by zero.) Therefore, the point \( (\beta_x, \gamma_x, \delta_x) \) cannot be a local maximum of \( \eta_x(\beta, \tilde{\gamma}_x(\beta), \delta_x) \) if \( \beta_x = 1 - \alpha_x \) and \( \frac{1}{2} \leq \alpha_x < 1 \).

Thus, the point \( (\beta_x, \gamma_x, \delta_x) \) lies in the interior of the domain \( K_x \), and hence it satisfies the system of equations \( \frac{\partial \eta}{\partial \beta} = \frac{\partial \eta}{\partial \gamma} = \frac{\partial \eta}{\partial \delta} = 0 \). Define \( \tau_x = \frac{1 - \alpha_x - \beta_x}{1 - \alpha_x - \beta_x - \gamma_x} \). From \( \frac{\partial \eta}{\partial \gamma} = 0 \) we find that

\[
\tau_x - 1 = \frac{\gamma_x}{1 - \alpha_x - \beta_x - \gamma_x} = \frac{\kappa_3(\beta_x - \gamma_x)}{\gamma_x},
\]

which gives us \( \beta_x = \frac{(\tau_x - 1) \kappa_3}{\gamma_x} + 1 \). Substituting this back into the expression for \( \tau_x \) and solving with respect to \( \gamma_x \), we obtain

\[
\gamma_x = (1 - \alpha_x) \frac{\tau_x - 1}{\kappa_3 (\tau_x - 1)^2 + 2 \tau_x - 1}.
\]

Next, since \( \frac{\partial \eta}{\partial \beta} = 0 \) and \( \frac{\partial \eta}{\partial \delta} = 0 \), we have

\[
\delta_x = \frac{\kappa_4(\beta_x - \gamma_x)(1 - \alpha_x - \beta_x)}{\kappa_2(1 - \alpha_x - \beta_x - \gamma_x)} = \frac{\kappa_4}{\kappa_2} \tau_x (\tau_x - 1) \gamma_x = (1 - \alpha_x) \frac{\kappa_4 (\tau_x - 1)^2}{\kappa_2 \kappa_3 (\frac{1}{\kappa_3} (\tau_x - 1)^2 + 2 \tau_x - 1)}.
\]

Recalling the definitions of \( p_x \) and \( q_x \), we can write

\[
\beta_x = (1 - \alpha_x) p_x, \quad \delta_x = (1 - \alpha_x) q_x.
\]

Substitute these expressions into the equation \( \frac{\partial \eta}{\partial \delta} = 0 \) to obtain

\[
(1 - \alpha_x)^{-1} = 1 + p_x + q_x + \frac{(s - 3) q_x}{2 \kappa_4} \pm \sqrt{\frac{(s - 2) q_x (p_x + q_x)}{\kappa_4} + \frac{(s - 3)^2 q_x^2}{4 \kappa_4}}.
\]

Since \( \beta_x + \delta_x \leq \alpha_x \), which is equivalent to \( (1 - \alpha_x)^{-1} \geq 1 + p_x + q_x \), we must take the positive sign outside of the radical. Therefore, it follows that

\[
(1 - \alpha_x)^{-1} = (1 - \alpha_x)^{-1}.
\]
By monotonicity of $\alpha$, we conclude that $\tau = \tau$, and hence $(\beta, \gamma, \delta) = (\beta, \gamma, \delta)$. Therefore, the point $(\beta, \gamma, \delta)$ is the only point where the global maximum of $\eta$ is attained on $K$. This completes the proof of Lemma A.1.

### A.2 Proof of Lemma A.3

We will use (A.10). Observe that the factor multiplying $\frac{\alpha'}{\alpha}$ in (A.10) can be bounded above by

$$2s - 4 - \frac{(s - 3)(s - 4)}{s - 2} + \frac{s(s - 1)}{rs - r - 2s} \leq s + 2$$  \hspace{1cm} (A.21)

when $r \geq s + 1 \geq 4$.

We now work towards an upper bound on $\alpha'$. Write $\alpha = \frac{N'}{1 + N}$. where

$$N = p + q + \frac{s - 3}{2\kappa}q = \sqrt{s - 2\kappa q(p + q) + \frac{(s - 3)^2}{4\kappa^2}q^2}.$$  

Then

$$\frac{\alpha'}{\alpha} = \frac{N'}{N(1 + N)}.$$  \hspace{1cm} (A.22)

It is not difficult to check that $1 \leq \kappa \leq 2$, which implies that

$$1 - \frac{2}{\tau} \leq p \leq 1, \quad q \geq \frac{\kappa}{\kappa^2} (\tau - 4),$$

$$p' = \frac{\kappa}{(\tau^2 + (\kappa - 1)(\tau - 1))^2} \leq \frac{2}{\tau^2},$$

$$q' = \frac{\kappa^3 (\tau - 1)(\tau - 1)^3 + \kappa(4\tau^2 - 3\tau + 1)}{\kappa^2 ((\tau - 1)^2 + \kappa(\tau - 1)^2)^2} \leq \frac{\kappa}{\kappa^2}.$$  

Recall that $\mu = 2\kappa + s - 3$. Applying (A.7) and the above inequalities, we have

$$N \geq p + 2q + \frac{s - 3}{\kappa^2}q \geq 1 - \frac{2}{\tau} + \frac{\mu(\tau - 4)}{\kappa^2} = \frac{\mu}{\kappa} \left(1 + \frac{\kappa}{\mu(\tau - 4)} - \frac{2}{\mu^2}\right)$$

and

$$N' \leq p' + q' + \frac{s - 3}{\kappa^2}q' \leq \frac{\mu}{\mu}$$

$$= p + 2q + \frac{s - 3}{\kappa^2}q + \frac{(s - 2)p' + q'p + 2q'p'}{\mu} + \frac{(s - 3)^2}{\kappa^2}q^2.$$
\[\leq \frac{2}{\tau^2} + \frac{\mu}{\kappa_2} + \frac{2(s-2)}{\mu\tau^2} + \frac{2\kappa_4(1 - \kappa_4)}{\kappa_2\mu} + \frac{s-2}{\mu(\tau - 4)}\]
\[= \frac{\mu}{\kappa_2} \left( 1 + \frac{2\kappa_4(1 - \kappa_4)}{\mu^2} + \frac{(s-2)\kappa_2}{\mu^2(\tau - 4)} + \frac{2\kappa_2}{\mu^2} + \frac{2(s-2)\kappa_2}{\mu^2\tau^2} \right).\]

Substituting these bounds into (A.22), we conclude that
\[
\frac{\alpha'_r}{\alpha_r} \leq \frac{\kappa_2}{\mu^2} \frac{\left( 1 + \frac{2\kappa_4(1 - \kappa_4)}{\mu^2} + \frac{(s-2)\kappa_2}{\mu^2(\tau - 4)} + \frac{2\kappa_2}{\mu^2} + \frac{2(s-2)\kappa_2}{\mu^2\tau^2} \right)}{\left( 1 + \frac{\kappa_2}{\mu^2} - \frac{4}{\tau} - \frac{2\kappa_2}{\mu^2\tau^2} \right) \left( 1 + \frac{2\kappa_2}{\mu^2} - \frac{4}{\tau} - \frac{2\kappa_2}{\mu^2\tau^2} \right)}.
\]

(A.23)

Using (5.7), we claim that whenever \(s \geq 3\) and \(r > \rho(s)\), we have
\[
\frac{1}{2} \leq \kappa_4 \leq 1 \text{ and } \kappa_4(1 - \kappa_4) \leq \frac{1}{2},
\]
\[
s - 2 \leq \mu \leq s - 1,
\]
\[
s^2 - s \leq \kappa_2 \leq s^2 + s - 3,
\]
\[
s \leq \frac{\kappa_2}{\mu} \leq \begin{cases} 8 & \text{if } s = 3, \\ s + 3 & \text{if } s \geq 4. \end{cases}
\]

For almost all these inequalities, the weak bound \(r \geq s + 1\) is sufficient. The bounds on \(\kappa_4\) are clear, and lead immediately to the bounds on \(\mu\). Use (A.19) for the bounds on \(\kappa_2\), with \(r - 2 \geq s - 1\). The lower bound on \(\kappa_2/\mu\) then follows, while for the upper bound we must be a little more precise. If \(s \geq 5\) then we bound \(r - 2 \geq s - 1\) in the definition of \(\kappa_2\), giving
\[
\frac{\kappa_2}{\mu} \leq \frac{s^2 + s - 3}{s - 1 - \frac{2}{s-1}} \leq s + 3.
\]

If \(s = 4\) then we have \(r > \rho(4) > 5\), so \(\mu \geq \frac{5}{2}\) and
\[
\frac{\kappa_2}{\mu} \leq \frac{34}{5} < 7 = s + 3.
\]

Finally, if \(s = 3\) then \(r \geq 4\) and \(\mu \geq 1\), while (A.19) implies that \(\kappa_2 \leq 8\).

First suppose that \(s \geq 4\). We prove (i) and (ii) at the same time (and indeed, prove that the condition on \((r, s)\) in (ii) is unnecessary when \(s \geq 4\)) by showing that \(\varphi'_a(x_\tau) < 0\) whenever \(\tau \geq \min\{2(s + 2)^2, (s + 1)^{2.5}\}\). Using the inequalities proved above, the denominator of (A.23) is bounded below by
\[
\left( 1 + \frac{\kappa_2}{\mu^2} - \frac{4}{\tau} - \frac{2\kappa_2}{\mu^2\tau^2} \right) \left( 1 + \frac{2\kappa_2}{\mu^2} - \frac{4}{\tau^2} - \frac{2\kappa_2}{\mu^2\tau^2} \right) \geq \left( 1 + \frac{s - 4}{\tau^2} - \frac{2s}{\tau^2} \right) \left( 1 + \frac{2(s - 2)}{\tau^2} - \frac{2s}{\tau^2} \right)
\]
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\[ \geq 1 + \frac{3s - 8}{\tau} + \frac{2(s^2 - 8s + 8)}{\tau^2} - \frac{2s(3s - 8)}{\tau^3}. \]

If \( s \geq 7 \) then \( s^2 - 8s + 8 \geq 0 \) and the denominator of (A.23) is bounded below by 1, since \( \tau^2 - 2s \geq 0 \) whenever \( \tau \geq \min\{2(s + 2)^2, (s + 1)^2.5\} \). For \( s = 4, 5, 6 \), direct substitution into the above expression confirms that the denominator of (A.23) is bounded below by 1.

Using this, we may apply our inequalities to the numerator of (A.23) to obtain

\[ \frac{\alpha'_r}{\alpha'_x} \leq \frac{s + 3}{\tau^2} \left( 1 + \frac{1}{2(s - 2)^2} + \frac{s + 3}{\tau - 4} + \frac{4(s + 3)}{\tau^2} \right). \]

Substituting this and (A.21) into (A.10), we find that \( \varphi'_\alpha(x, \tau) < 0 \) if

\[ (s + 2)(s + 3) \left( 1 - \frac{1}{2(s - 2)^2} + \frac{s + 3}{\tau - 4} + \frac{4(s + 3)}{\tau^2} \right) < \tau. \]  

(A.24)

If \( s \geq 5 \) then the left hand side of (A.24) is bounded above by

\[ (s + 2)(s + 3) \left( 1 + \frac{1}{18} + \frac{8}{84} + \frac{32}{88^2} \right) = \frac{8804}{7623} (s + 2)(s + 3), \]

which is bounded above by \( \min\{2(s + 2)^2, (s + 1)^2.5\} \). Hence (i) and (ii) hold whenever \( s \geq 5 \) and \( r > \rho(s) \).

When \( s = 4 \) we have \( r > \rho(4) > 5 \) and \( \min\{2(s + 2)^2, (s + 1)^2.5\} = 52.5 > 55 \). We bound the left hand side of (A.21) by 26/5. Substituting this into (A.21), we see that when \( \tau \geq 55 \), the left hand side of (A.24) is bounded above by

\[ \frac{7 \times 26}{4} \left( 1 + \frac{1}{8} + \frac{7}{51} + \frac{28}{55} \right) < 55. \]

This implies that (i) and (ii) hold whenever \( s = 4 \) and \( r > \rho(4) \).

For the final case, suppose that \( s = 3 \). Then we must use the bound \( \frac{\alpha'_r}{\mu} \leq 8 \), so the numerator of (A.23) becomes

\[ 8 \left( \frac{3}{2} + \frac{8}{\tau - 4} + \frac{32}{\tau^2} \right). \]  

(A.25)

For (ii) we also assume that \( 2(r - 1) \geq 4^3 \), which means that \( r \geq 33 \). Hence the left hand side of (A.21) is bounded above by \( \frac{21}{10} \), and for all \( \tau \geq 32 \) the left hand side of (A.24) is bounded above by

\[ 8 \times \frac{21}{10} \left( \frac{3}{2} + \frac{8}{28} + \frac{1}{32} \right) \]

which is strictly less than 32, as required. This proves that (ii) holds when \( s = 3 \).

To prove (i) when \( s = 3 \), assume that \( r \geq 4 \). The left hand side of (A.21) is bounded above by \( 2 + \frac{3}{r - 3} \), and the numerator of (A.23) is bounded above by the expression given in (A.25). Therefore a sufficient condition for \( \varphi'_\alpha(x, \tau) \), when \( s = 3 \), is

\[ 8 \left( 2 + \frac{3}{r - 3} \right) \left( \frac{3}{2} + \frac{8}{\tau - 4} + \frac{32}{\tau^2} \right) < \tau. \]
If $r \geq 5$ then this inequality holds for all $\tau \geq 50$. Therefore (i) holds when $s = 3$ and $r \geq 5$.

Finally, suppose that $s = 3$ and $r = 4$. Then

$$\kappa_4 = \frac{1}{2}, \quad \mu = 1, \quad \kappa_2 = 8.$$  

Substituting these values into (A.10), we find that $\varphi'(x, \tau) < 0$ if

$$60 \left( 1 + \frac{4}{\tau-4} + \frac{16}{\tau^2} \right) < \tau \left( 1 + \frac{16}{\tau} + \frac{16}{\tau^2} - \frac{256}{\tau^3} \right).$$

This inequality holds whenever $\tau \geq 50$, completing the proof that (i) holds whenever $s = 3$ and $r = 4$. This completes the proof of Lemma A.3. \hfill \Box