Basic qualitative and quantitative results for solutions to nonlinear, dynamic equations on time scales with an application to economic modelling

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Abstract

This article investigates both basic qualitative and basic quantitative properties of solutions to first– and higher–order dynamic equations on time scales and thus provides a foundation and framework for future advanced nonlinear studies in the field. Particular focus lies in the: existence; uniqueness; dependency; approximation; and explicit representation, of solutions to nonlinear initial value problems. The main tools used are from modern areas of nonlinear analysis, including: the fixed–point theorems of Banach and Schäfer; the method of successive approximations; a novel definition of measuring distance in metric spaces and normed spaces; and a "separation" of variables technique is introduced to the general time scale setting.

The new results compliment and extend those of Stefan Hilger's seminal paper of 1990.

As an application of the new results we present and analyse a simple model from economics, known as the Keynesian–Cross model with "lagged" income, in the general time scale environment.

Ideas suggesting further applications and possible new directions for the novel results are also presented.

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1 Introduction

Important advancements in all the physical–, life– and social–sciences rest heavily on the existence of a mathematical framework to describe, to solve and to better understand the problems from these fields. Historically, two separate approaches have dominated mathematical modelling: the field of differential equations, termed "continuous dynamic modelling", where variables (eg. time) are assumed to flow in a continuous fashion; and the area of difference equations, termed "discrete dynamic modelling", where variables (such as time) are assumed to vary in a discrete manner.

Traditionally, researchers have assumed that dynamical processes are *either* continuous *or* discrete and thus have employed *either* differential equations *or* difference equations - but not elements from both schools of thought - for the mathematical description and analysis of dynamic models. For example, the classical approach of Domar [3] used differential equations to analyse an expanding economy, while, on the other hand, Harrod's method [3] of mathematical description for economic growth involved the field of difference equations.

This blanket assumption that processes are either solely continuous or solely discrete, while convenient for traditional mathematical approaches, is flawed because, in reality, many processes do feature both continuous and discrete elements. Thus, traditional mathematical modelling techniques, such as differential or difference equations, provide a limited understanding of these types of physical models and appears to be a case of modifying the assumptions on a physical problem to best fit the mathematics, rather than vice–versa.

In particular, certain economically important phenomena do not possess solely continuous properties or solely discrete aspects. Rather, these phenomena contain processes that feature elements of *both* the continuous *and* the discrete. A simple example of this hybrid continuous–discrete behaviour is seen in "seasonally breeding populations in which generations do not overlap. Many natural populations, particularly among temperate–zone insects (including many economically important crop and orchid pests) are of this kind" [30, p.460]. These insects lay their eggs just before the generation dies out at the end of the season, with the eggs laying dormant, hatching at the start of the next season giving rise to a new, nonoverlapping generation. The continuous–discrete behaviour in seen in the fact that during each generation the population varies continuously (due to mortality, resource consumption, predation, interaction etc.), while the population varies in a discrete fashion between the end of one generation and the beginning of the next [17, p.620].

In addition, continuous–discrete processes are seen in: robust 3D tracking in shape and motion estimation [31, p.712]; option–pricing and stock dynamics in finance [6, p.3]; the frequency of markets and duration of market trading in economics [24, p.1], [28, p.2]; large–scale models of DNA dynamics [26, p.2504]; gene mutation fixation [9, pp.1–2]; and "hybrid systems" where stop–start elements are naturally seen.

Current approaches, such as the field of differential equations or the field of difference equations are ill–equipped as separate fields to accurately describe the above models because these mathematical areas are limited to either the continuous or the discrete and thus are of limited value in understanding these models. Therefore, there is a great need to find a more flexible mathematical framework to accurately model the aforementioned dynamical blend of systems so that they are precisely described, better–understood and significant advancements are made.

To address the aforementioned needs, an emerging, progressive and modern area of mathematics, known as the field of *dynamic equations on time scales*, has the capacity to act as the framework to effectively describe the above phenomena and to make advancements in their associated fields. Created by Hilger in 1990 [23] and developed by others (see [5, 27] and references therein), this new and exciting type of mathematics is more general and versatile than the traditional theories of differential and difference equations as it can, under one framework, mathematically describe continuous–discrete hybrid processes and hence is the optimal way forward for accurate and malleable mathematical modelling. In fact, the progressive field of dynamic equations on time scales contains, links and extends the classical theory of differential and difference equations.

Much of the “linear” theory of dynamic equations on time scales has been presented in [5], however, there is significantly less literature available on the basic “nonlinear” theory of the field. It is important to bridge this gap between known linear studies and unknown nonlinear theory, as the processes in our world are inherently nonlinear and such investigations will provide an important platform for gaining a deeper understanding of our environment.

This paper considers first–order dynamic equations of the type

$$\mathbf{x}^{\Delta} = \mathbf{f}(t, \mathbf{x}), \qquad t \in [a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}; \tag{1.1}$$

$$\mathbf{x}^{\Delta} = \mathbf{f}(t, \mathbf{x}^{\sigma}), \qquad t \in [a, b]_{\mathbb{T}}; \tag{1.2}$$

subject to the initial condition

$$\mathbf{x}(a) = \mathbf{A}; \tag{1.3}$$

where $\mathbf{f} : [a, b]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ may be a nonlinear function, $n \geq 1$; t is from a so–called “time scale” $\mathbb{T}$ (which is a nonempty closed subset of $\mathbb{R}$); $\mathbf{x}^{\Delta}$ is the generalised “delta” derivative of $\mathbf{x}$; and $a < b$ are given constants in $\mathbb{T}$; and $\mathbf{A}$ is a given constant in $\mathbb{R}^n$. Equation (1.1) subject to (1.3) is known as a dynamic initial value problem (IVP) on time scales. Equations (1.2), (1.3) are defined similarly, where $\mathbf{x}^{\sigma} := \mathbf{x} \circ \sigma$ with σ a function to be defined a little later.

We will also consider the generalised form of (1.1) and (1.2), namely

$$\mathbf{x}^{\Delta} = \mathbf{f}(t, \mathbf{x}, \mathbf{x}^{\sigma}), \quad t \in [a, b]_{\mathbb{T}};$$

and higher–order dynamic equations of the type

$$x^{\Delta^n} = f(t, x, x^{\Delta}, x^{\Delta^2}, \ldots, x^{\Delta^{n-1}}), \quad t \in [a, b]_{\mathbb{T}}.$$

If $\mathbb{T} = \mathbb{R}$ then $\mathbf{x}^{\Delta} = \mathbf{x}'$ and (1.1) becomes the familiar ordinary differential equation

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}), \quad t \in [a, b].$$

If $\mathbb{T} = \mathbb{Z}$ then $\mathbf{x}^{\Delta} = \mathbf{x}(t + 1) - \mathbf{x}(t)$ and (1.1) becomes the well–known difference equation

$$\Delta\mathbf{x} := \mathbf{x}(t + 1) - \mathbf{x}(t) = \mathbf{f}(t, \mathbf{x}), \quad t \in \{a, a + 1, \ldots, b\}.$$

There are many more time scales than just $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ and hence many more dynamic equations.

This paper focuses on the qualitative and quantitative properties of solutions to dynamic equations on time scales. Some important questions that this work addresses are:

- Under what conditions do the above dynamic equations actually have (possibly unique) solutions?
- If solutions do exist, then what are their nature; and how can we find them; or closely approximate them?

The main tools that we use to answer the above questions are from modern areas of nonlinear analysis, including: the fixed–point theorems of Banach and Schäfer; the method of successive approximations; and a novel definition of measuring distance in metric spaces and normed spaces. In addition, a new "separation" of variables technique is introduced to the general time scale setting.

The results contained herein compliment and extend those of Stefan Hilger's seminal paper of 1990 [23] and provide a new foundation and framework for future advanced nonlinear studies in the field.

As an application of our new ideas, we present and analyse a simple model from economics in the general time scale setting. Moreover, further applications and possible new directions for the novel results are also presented.

To understand the notation used above, some preliminary definitions are needed, which are now presented. For more detail see [5, Chap.1] or [23].

Definition 1.1 *A time scale* $\mathbb{T}$ *is a nonempty closed subset of the real numbers* $\mathbb{R}$.

Since a time scale may or may not be connected, the concept of the jump operator is useful to define the generalised derivative $\mathbf{x}^{\Delta}$ of a function $\mathbf{x}$.

Definition 1.2 *The forward (backward) jump operator* $\sigma(t)$ *at* t *for* $t < \sup \mathbb{T}$ *(respectively* $\rho(t)$ *at* t *for* $t > \inf \mathbb{T}$*) is given by*

$$\sigma(t) := \inf\{\tau > t : \tau \in \mathbb{T}\}, \quad (\rho(t) := \sup\{\tau < t : \tau \in \mathbb{T}\},) \quad \text{for all } t \in \mathbb{T}.$$

Define the graininess function $\mu : \mathbb{T} \to [0, \infty)$ *as* $\mu(t) := \sigma(t) - t$.

Throughout this work the assumption is made that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$.

Definition 1.3 *The jump operators* σ *and* ρ *allow the classification of points in a time scale in the following way: If* $\sigma(t) > t$*, then the point* t *is called right–scattered; while if* $\rho(t) < t$*, then* t *is termed left–scattered. If* $t < \sup \mathbb{T}$ *and* $\sigma(t) = t$*, then the point* t *is called right–dense; while if* $t > \inf \mathbb{T}$ *and* $\rho(t) = t$*, then we say* t *is left–dense.*

If $\mathbb{T}$ has a left–scattered maximum value m, then we define $\mathbb{T}^{\kappa} := \mathbb{T} - \{m\}$. Otherwise $\mathbb{T}^{\kappa} := \mathbb{T}$.

The following gives a formal $\varepsilon - \delta$ definition of the generalised delta derivative.

Definition 1.4 *Fix* $t \in \mathbb{T}^{\kappa}$ *and let* $\mathbf{x} : \mathbb{T} \to \mathbb{R}^n$*. Define* $\mathbf{x}^{\Delta}(t)$ *to be the vector (if it exists) with the property that given* $\varepsilon > 0$ *there is a neighbourhood* U *of* t *with*

$$|[x_i(\sigma(t)) - x_i(s)] - x_i^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|, \quad \text{for all } s \in U \text{ and each } i = 1, \ldots, n.$$

Call $\mathbf{x}^{\Delta}(t)$ *the delta derivative of* $\mathbf{x}(t)$ *and say that* $\mathbf{x}$ *is delta–differentiable.*

Converse to the delta derivative, we now state the definition of delta integration.

Definition 1.5 *If* $\mathbf{K}^{\Delta}(t) = \mathbf{k}(t)$ *then define the delta integral by*

$$\int_a^t \mathbf{k}(s)\Delta s = \mathbf{K}(t) - \mathbf{K}(a).$$

If $\mathbb{T} = \mathbb{R}$ *then* $\int_a^t \mathbf{k}(s)\Delta s = \int_a^t \mathbf{k}(s)ds$*, while if* $\mathbb{T} = \mathbb{Z}$ *then* $\int_a^t \mathbf{k}(s)\Delta s = \sum_a^{t-1} \mathbf{k}(s)$*. Once again, there are many more time scales than just* $\mathbb{R}$ *and* $\mathbb{Z}$ *and hence there are many more delta integrals. For a more general definition of the delta integral see [5].*

The following theorem will be fundamental.

Theorem 1.6 *[23] Assume that* $\mathbf{k} : \mathbb{T} \to \mathbb{R}^n$ *and let* $t \in \mathbb{T}^{\kappa}$*.*
(i) If $\mathbf{k}$ *is delta–differentiable at* t *then* $\mathbf{k}$ *is continuous at* t*.*
(ii) If $\mathbf{k}$ *is continuous at* t *and* t *is right–scattered then* $\mathbf{k}$ *is delta–differentiable at* t *with*

$$\mathbf{k}^{\Delta}(t) = \frac{\mathbf{k}(\sigma(t)) - \mathbf{k}(t)}{\sigma(t) - t}.$$

(iii) If $\mathbf{k}$ *is delta–differentiable and* t *is right–dense then*

$$\mathbf{k}^{\Delta}(t) = \lim_{s \to t} \frac{\mathbf{k}(t) - \mathbf{k}(s)}{t - s}.$$

(iv) If $\mathbf{k}$ *is delta–differentiable at* t *then* $\mathbf{k}(\sigma(t)) = \mathbf{k}(t) + \mu(t)\mathbf{k}^{\Delta}(t)$*.*

For brevity, we will write $\mathbf{x}^{\sigma}$ to denote the composition $\mathbf{x} \circ \sigma$.
The following gives a generalised idea of continuity on time scales.

Definition 1.7 *Assume* $\mathbf{k} : \mathbb{T} \to \mathbb{R}^n$*. Define and denote* $\mathbf{k} \in C_{rd}(\mathbb{T}; \mathbb{R}^n)$ *as right–dense continuous (rd–continuous) if:* $\mathbf{k}$ *is continuous at every right–dense point* $t \in \mathbb{T}$*; and* $\lim_{s \to t^-} \mathbf{k}(s)$ *exists and is finite at every left–dense point* $t \in \mathbb{T}$*.*

Of particular importance is the fact that every C_{rd} function is delta–integrable [5, Theorem 1.73].

A solution to (1.1), (1.3) is a delta–differentiable function $\mathbf{x} : [a, \sigma(b)]_{\mathbb{T}} \to \mathbb{R}^n$ that satisfies (1.1) and (1.3). A solution to (1.2), (1.3) is defined similarly.

Throughout this work, if $\mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ then $\langle \mathbf{y}, \mathbf{z} \rangle$ denotes the usual Euclidean inner product on $\mathbb{R}^n$ and $\|\mathbf{z}\|$ denotes the Euclidean norm of $\mathbf{z}$ on $\mathbb{R}^n$.

For more on the basic theory of time scales, see [1, 2, 5, 7, 8, 11, 14, 15, 19, 20, 21, 22, 23, 25, 27, 37, 33, 34, 35, 38, 39, 40].

2 Preliminary lemmas

In this section we present some basic lemmas in which we reformulate our dynamic equations as equivalent delta integral equations. The approach is based on the ideas in [23] and will be of fundamental importance in following sections.

Lemma 2.1 *Consider* (1.1), (1.3). *Let* $\mathbf{f} : [a,b]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ *be continuous.*

(i) *If* $\mathbf{x} \in C([a,\sigma(b)]_{\mathbb{T}};\mathbb{R}^n)$ *is a solution of* (1.1), (1.3) *then*

$$\mathbf{x}(t) = \int_a^t \mathbf{f}(s,\mathbf{x}(s))\ \Delta s + \mathbf{A}, \quad t \in [a,\sigma(b)]_{\mathbb{T}}; \tag{2.1}$$

(ii) *If* $\mathbf{x} \in C([a,\sigma(b)]_{\mathbb{T}};\mathbb{R}^n)$ *satisfies* (2.1) *then* $\mathbf{x}^{\Delta} \in C([a,b]_{\mathbb{T}};\mathbb{R}^n)$ *and* $\mathbf{x}$ *is a solution of* (1.1), (1.3).

Similarly, we have the following result.

Lemma 2.2 *Consider* (1.2), (1.3). *Let* $\mathbf{f} : [a,b]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ *be continuous.*

(i) *If* $\mathbf{x} \in C([a,\sigma(b)]_{\mathbb{T}};\mathbb{R}^n)$ *is a solution of* (1.2), (1.3) *then*

$$\mathbf{x}(t) = \int_a^t \mathbf{f}(s,\mathbf{x}^{\sigma}(s))\ \Delta s + \mathbf{A}, \quad t \in [a,\sigma(b)]_{\mathbb{T}}; \tag{2.2}$$

(ii) *If* $\mathbf{x} \in C([a,\sigma(b)]_{\mathbb{T}};\mathbb{R}^n)$ *satisfies* (2.2) *then* $\mathbf{x}^{\Delta} \in C_{rd}([a,b]_{\mathbb{T}};\mathbb{R}^n)$ *and* $\mathbf{x}$ *is a solution of* (1.2), (1.3).

We will also consider (1.1) and (1.2) with a right–dense continuous right–hand side. This definition generalises the idea of Definition 1.7 and is weaker than the usual assumption of continuity. The mapping $\mathbf{f} : [a,b]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ is called right–dense continuous if: $\mathbf{f}$ is continuous at each $(t,\mathbf{x})$ where t is right–dense; and the limits

$$\lim_{(s,\mathbf{y})\to(t^-,\mathbf{x})} \mathbf{f}(s,\mathbf{y}) \quad \text{and} \quad \lim_{\mathbf{y}\to\mathbf{x}} \mathbf{f}(t,\mathbf{y})$$

both exist (and are finite) at each $(t,\mathbf{x})$ where t is left–dense.

Lemma 2.3 *Consider* (1.1), (1.3). *Let* $\mathbf{f} : [a,b]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ *be rd–continuous. We then have* **(i)** *and* **(ii)** *of Lemma 2.1 holding with "*$\mathbf{x}^{\Delta} \in C([a,b]_{\mathbb{T}};\mathbb{R}^n)$*" replaced by "*$\mathbf{x}^{\Delta} \in C_{rd}([a,b]_{\mathbb{T}};\mathbb{R}^n)$*".*

Lemma 2.4 *Consider* (1.2), (1.3). *Let* $\mathbf{f} : [a,b]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ *be rd–continuous. We then have* **(i)** *and* **(ii)** *of Lemma 2.2 holding.*

The proofs of all of the above results are straightforward and so are omitted. We remark that (2.1) and (2.2) are well–defined as continuous or rd–continuous functions are always delta integrable. We also remark that σ is, in general, a right–dense continuous function and that the differences in the continuity of $\mathbf{x}^{\Delta}$ in the above theorems are a result of the fact that the compositions of continuous and rd–continuous functions are always rd–continuous,

3 Contractive mapping approach

In this section we obtain some new results concerning the existence, uniqueness, dependency and approximation of solutions to the dynamic IVPs: (1.1), (1.3); and (1.2), (1.3). The ideas will rely on Banach's fixed–point theorem and a novel definition of measuring distance in metric spaces (and normed spaces).

Banach's fixed–point theorem is one of the simplest, yet most powerful, ideas from fixed–point theory - mainly because the theorem produces a wide range of qualitative *and* quantitative information about solutions. For example, the theorem's basic idea of successive approximations can be utilised, via a computer, to find the fixed–point of a contractive map; and can yield approximations to any degree of precision. In addition, the number of iterations required to obtain a desired accuracy can be ascertained [12, p.9], [36, pp.2–3].

Let (Y, d) be a complete metric space and $F : Y \to Y$. The map F is said to be contractive if there exists a positive constant $\alpha < 1$ such that

$$d(F(x), F(y)) \leq \alpha d(x, y), \quad \forall x, y \in Y.$$

The constant α is called the contraction constant of F.

For any given $y \in Y$ we define the sequence $\{F^i(y)\}$ recursively by: $F^0(y) := y$; and $F^{i+1}(y) := F(F^i(y))$.

Theorem 3.1 (Banach, [12] p.10) *Let (Y, d) be a complete metric space and let $F : Y \to Y$ be contractive. Then F has a unique fixed–point u and $F^i(y) \to u$ for each $y \in Y$.*

Remark 3.2 *It is well–known [12, p.10] that if we start at an arbitrary $y \in Y$ then Banach's theorem provides the following estimate on the "error" between the ith iteration $F^i y$ and the fixed point u, namely*

$$d(F^i y, u) \leq \frac{\alpha^i}{1-\alpha} d(y, Fy). \tag{3.1}$$

An important question regarding the contraction condition on F in Banach's fixed–point theorem is: what is a suitable metric to define on Y so that we can "maximise" the class of F that will be contractive, with the minimum amount of conditions? For example, a map may not be contractive under one particular definition of metric, however, the same map may be contractive with respect to a different metric [12, pp.24–25].

With the above question in mind, we now introduce a novel metric (and norm) in the time scale setting with the ideas involving the generalised exponential function on a time scale. For this, we require a few more definitions to assist with our investigation.

Define the so–called set of regressive functions, $\mathcal{R}$, by

$$\mathcal{R} := \{p \in C_{rd}(\mathbb{T}; \mathbb{R}) \text{ and } 1 + p(t)\mu(t) \neq 0, \ \forall t \in \mathbb{T}\}$$

and the set of positively regressive functions, $\mathcal{R}^+$, by

$$\mathcal{R}^+ := \{p \in C_{rd}(\mathbb{T}; \mathbb{R}) \text{ and } 1 + p(t)\mu(t) > 0, \ \forall t \in \mathbb{T}\}. \tag{3.2}$$

For $p \in \mathcal{R}$ we define (see [5, Theorem 2.35]) the exponential function $e_p(\cdot, t_0)$ on the time scale $\mathbb{T}$ as the unique solution to the scalar IVP

$$x^\Delta = p(t)x, \quad x(t_0) = 1.$$

If $p \in \mathcal{R}^+$ then $e_p(t, t_0) > 0$ for all $t \in \mathbb{T}$, [5, Theorem 2.48].

More explicitly, the exponential function $e_p(\cdot, t_0)$ is given by

$$e_p(t, t_0) := \begin{cases} \exp\left(\int_{t_0}^t p(s)\, ds\right), & \text{for } t \in \mathbb{T},\ \mu = 0; \\ \exp\left(\int_{t_0}^t \dfrac{\operatorname{Log}(1 + \mu(s)p(s))}{\mu(s)}\, \Delta s\right), & \text{for } t \in \mathbb{T},\ \mu > 0; \end{cases}$$

where Log is the principal logarithm function.

Let $\beta > 0$ be a constant and let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^n$. We will consider the space of continuous functions $C([a, \sigma(b)]_{\mathbb{T}}; \mathbb{R}^n)$ coupled with a suitable metric, either

$$d_\beta(\mathbf{x}, \mathbf{y}) := \sup_{t \in [a, \sigma(b)]_{\mathbb{T}}} \frac{\|\mathbf{x}(t) - \mathbf{y}(t)\|}{e_\beta(t, a)}, \text{ which we term the “TZ–metric”;}$$

or

$$d_0(\mathbf{x}, \mathbf{y}) := \sup_{t \in [a, \sigma(b)]_{\mathbb{T}}} \|\mathbf{x}(t) - \mathbf{y}(t)\|, \quad \text{the well–known sup–metric.}$$

We will also consider $C([a, \sigma(b)]_{\mathbb{T}}; \mathbb{R}^n)$ coupled with a suitable norm, either

$$\|\mathbf{x}\|_\beta := \sup_{t \in [a, \sigma(b)]_{\mathbb{T}}} \frac{\|\mathbf{x}(t)\|}{e_\beta(t, a)}, \quad \text{which we term the “TZ–norm”;}$$

or

$$\|\mathbf{x}\|_0 := \sup_{t \in [a, \sigma(b)]_{\mathbb{T}}} \|\mathbf{x}(t)\|, \quad \text{the well–known sup–norm.}$$

The above definitions of d_β and $\|\cdot\|_\beta$ are new generalisations of Bielecki’s metric and norm [4], [12, pp.25–26], [13, pp.153–155], [36, p.44] in the time scale environment.

Some important properties of d_β and $\|\cdot\|_\beta$ are now listed.

Lemma 3.3 *If $\beta > 0$ is a constant then:*

(i) *d_β is a metric;*

(ii) *$(C([a, \sigma(b)]_{\mathbb{T}}; \mathbb{R}^n), d_\beta)$ is a complete metric space;*

(iii) *$\|\cdot\|_\beta$ is a norm and is equivalent to the sup–norm $\|\cdot\|_0$;*

(iv) *$(C([a, \sigma(b)]_{\mathbb{T}}; \mathbb{R}^n), \|\cdot\|_\beta)$ is a Banach space.*

Proofs

(i) If $\beta > 0$ is a constant then from (3.2) we have $\beta \in \mathcal{R}^+$. Thus, we have $e_\beta(t, a) > 0$ for all $t \in [a, \sigma(b)]_{\mathbb{T}}$ from [5, Theorem 2.48]. The three properties of a metric [10, p.21] are now easily verified.

(ii) Let $\{\mathbf{x}_i(t)\}$ be a Cauchy sequence, that is, for every $\varepsilon > 0$ there exists a positive integer N_ε such that

$$\frac{\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\|}{e_\beta(t, a)} < \varepsilon, \quad \forall i, j > N_\varepsilon, \quad \forall t \in [a, \sigma(b)]_{\mathbb{T}}.$$

It follows that the sequence $\{\mathbf{x}_i(t)\}$ is uniformly convergent. The limit of a uniformly convergent sequence of continuous functions is also a continuous function. Taking $j \to \infty$ above we have

$$\frac{\|\mathbf{x}_i(t) - \mathbf{x}(t)\|}{e_\beta(t,a)} < \varepsilon, \quad \forall i > N_\varepsilon, \quad \forall t \in [a, \sigma(b)]_{\mathbb{T}}$$

and thus the Cauchy sequence $\{\mathbf{x}_i(t)\}$ converges in the metric d_β of $C([a,\sigma(b)]_{\mathbb{T}}; \mathbb{R}^n)$ to a function $\mathbf{x}(t) \in C([a,\sigma(b)]_{\mathbb{T}}; \mathbb{R}^n)$. Thus, $(C([a,\sigma(b)]_{\mathbb{T}}; \mathbb{R}^n), d_\beta)$ is a complete metric space.

(iii) From **(i)** it follows that $\|\cdot\|_\beta$ is a norm. We show that there exist positive constants k and K such that

$$k\|\mathbf{x}\|_0 \le \|\mathbf{x}\|_\beta \le K\|\mathbf{x}\|_0. \tag{3.3}$$

Since $\beta > 0$ we have $e_\beta(t,a) > 0$ for $t \in [a,\sigma(b)]_{\mathbb{T}}$. Hence,

$$[e_\beta(t,a)]^\Delta = \beta e_\beta(t,a) > 0, \quad t \in [a,\sigma(b)]_{\mathbb{T}};$$

so that $1/e_\beta(t,a)$ is strictly decreasing for $t \in [a,\sigma(b)]_{\mathbb{T}}$. We also have $e_\beta(a,a) = 1$ from [5, Theorem 2.36 (i)]. Combining the above ideas we have

$$\frac{\|\mathbf{x}\|_0}{e_\beta(\sigma(b),a)} \le \|\mathbf{x}\|_\beta \le 1\|\mathbf{x}\|_0;$$

so that (3.3) holds with $k := 1/e_\beta(\sigma(b),a)$ and $K := 1$. Hence the TZ–norm $\|\cdot\|_\beta$ and the sup–norm $\|\cdot\|_0$ are equivalent.

(iv) This follows from **(ii)** and **(iii)**. □

We are now ready to present the main result of this section, which will be proved by using Banach's theorem.

Theorem 3.4 *Consider the dynamic IVP* (1.1), (1.3). *Let* $\mathbf{f} : [a,b]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ *be rd–continuous and let* L *be a positive constant. If*

$$\|\mathbf{f}(t,\mathbf{p}) - \mathbf{f}(t,\mathbf{q})\| \le L\|\mathbf{p} - \mathbf{q}\|, \quad \forall t \in [a,b]_{\mathbb{T}},\ (\mathbf{p},\mathbf{q}) \in \mathbb{R}^{2n}; \tag{3.4}$$

then the dynamic IVP (1.1), (1.3) *has a unique solution. In addition, if a sequence of functions* $\{\mathbf{x}_i\}$ *is defined inductively by choosing any* $\mathbf{x}_0 \in C([a,\sigma(b)]_{\mathbb{T}}; \mathbb{R}^n)$ *and setting*

$$\mathbf{x}_{i+1}(t) = \mathbf{A} + \int_a^t \mathbf{f}(s, \mathbf{x}_i(s))\ \Delta s \tag{3.5}$$

then the sequence $\{\mathbf{x}_i\}$ *converges uniformly on* $[a,\sigma(b)]_{\mathbb{T}}$ *to the unique solution* $\mathbf{x}$ *of* (1.1), (1.3). *Furthermore,* $\mathbf{x}^\Delta \in C_{rd}([a,b]_{\mathbb{T}}; \mathbb{R}^n)$.

Proof Since $\mathbf{f}$ is a rd–continuous function, (3.5) is well–defined. Let $L > 0$ be the constant defined in (3.4) and let $\beta := L\gamma$ where $\gamma > 1$ is an arbitrary constant. Consider the complete metric space $(C([a,\sigma(b)]_{\mathbb{T}}; \mathbb{R}^n), d_\beta)$ and let

$$\mathbf{F} : C([a,\sigma(b)]_{\mathbb{T}}; \mathbb{R}^n) \to C([a,\sigma(b)]_{\mathbb{T}}; \mathbb{R}^n)$$

be defined by

$$[\mathbf{F}\mathbf{x}](t) := \int_a^t \mathbf{f}(s, \mathbf{x}(s))\ \Delta s + \mathbf{A}, \quad t \in [a,\sigma(b)]_{\mathbb{T}}. \tag{3.6}$$

By Lemma 2.3, fixed–points of $\mathbf{F}$ will be solutions to the dynamic IVP (1.1), (1.3). Thus, we want to prove that there exists a unique $\mathbf{x}$ such that $\mathbf{Fx} = \mathbf{x}$. To do this, we show that $\mathbf{F}$ is a contractive map with contraction constant $\alpha = 1/\gamma < 1$ and Banach's fixed–point theorem will then apply. For any $\mathbf{u}, \mathbf{v} \in C([a, \sigma(b)]_{\mathbb{T}}; \mathbb{R}^n)$, consider

$$\begin{aligned}
d_\beta(\mathbf{Fu}, \mathbf{Fv}) &:= \sup_{t\in[a,\sigma(b)]_{\mathbb{T}}} \frac{\|[\mathbf{Fu}](t) - [\mathbf{Fv}](t)\|}{e_\beta(t,a)} \\
&\leq \sup_{t\in[a,\sigma(b)]_{\mathbb{T}}} \left[\frac{1}{e_\beta(t,a)} \int_a^t \|\mathbf{f}(s, \mathbf{u}(s)) - \mathbf{f}(s, \mathbf{v}(s))\| \ \Delta s\right] \\
&\leq \sup_{t\in[a,\sigma(b)]_{\mathbb{T}}} \left[\frac{1}{e_\beta(t,a)} \int_a^t L\|\mathbf{u}(s) - \mathbf{v}(s)\| \ \Delta s\right], \quad \text{from (3.4)} \\
&= L \sup_{t\in[a,\sigma(b)]_{\mathbb{T}}} \left[\frac{1}{e_\beta(t,a)} \int_a^t e_\beta(s,a) \frac{\|\mathbf{u}(s) - \mathbf{v}(s)\|}{e_\beta(s,a)} \ \Delta s\right] \\
&\leq L d_\beta(\mathbf{u}, \mathbf{v}) \sup_{t\in[a,\sigma(b)]_{\mathbb{T}}} \left[\frac{1}{e_\beta(t,a)} \int_a^t e_\beta(s,a) \ \Delta s\right] \\
&= L d_\beta(\mathbf{u}, \mathbf{v}) \sup_{t\in[a,\sigma(b)]_{\mathbb{T}}} \left[\frac{1}{e_\beta(t,a)} \left(\frac{e_\beta(t,a) - 1}{\beta}\right)\right] \\
&= \frac{d_\beta(\mathbf{u}, \mathbf{v})}{\gamma} \sup_{t\in[a,\sigma(b)]_{\mathbb{T}}} \left[1 - \frac{1}{e_\beta(t,a)}\right], \quad \text{since } \beta = L\gamma, \\
&= \frac{d_\beta(\mathbf{u}, \mathbf{v})}{\gamma} \left[1 - \frac{1}{e_\beta(\sigma(b),a)}\right] < \frac{d_\beta(\mathbf{u}, \mathbf{v})}{\gamma}.
\end{aligned}$$

As $\gamma > 1$ we see that $\mathbf{F}$ is a contractive map and Banach's fixed–point theorem applies, yielding the existence of a unique fixed–point $\mathbf{x}$ of $\mathbf{F}$. In addition, from Banach's theorem, the sequence $\{\mathbf{x}_i\}$ defined in (3.5) converges uniformly in the TZ–norm $\|\cdot\|_\beta$ and thus the sequence $\{\mathbf{x}_i\}$ converges uniformly in the sup–norm $\|\cdot\|_0$ to that fixed–point $\mathbf{x}$. This completes the proof. □

Theorem 3.4 extends the ideas of [23, Theorem 5.5] where the condition

$$L[\sigma(b) - a] < 1 \tag{3.7}$$

was imposed. If $L[\sigma(b)-a] \geq 1$ then [23, Theorem 5.5] does not directly apply to the dynamic IVP (1.1), (1.3) and, moreover, the use of the TZ–metric d_β in the proof of Theorem 3.4 demonstrates that the assumption on $L[\sigma(b) - a]$ in (3.7) is removable from [23, Theorem 5.5].

In view of Remark 3.2, our approach in the proof of Theorem 3.4 can be used to evaluate the rate of convergence of iterates. If $\mathbf{x}$, $\mathbf{x_0} \in C([a, \sigma(b)]_{\mathbb{T}}; \mathbb{R}^n)$ then for $\beta := L\gamma$ with $\gamma > 1$ we have from (3.1)

$$d_\beta(\mathbf{F}^i\mathbf{x}_0, \mathbf{x}) \leq \frac{\gamma^{-i}}{1 - \gamma^{-1}} d_\beta(\mathbf{x}_0, \mathbf{F}\mathbf{x}_0)$$

and so

$$\|\mathbf{F}^i\mathbf{x}_0 - \mathbf{x}\| \leq e_\beta(\sigma(b), a) \frac{\gamma^{-i}}{1 - \gamma^{-1}} \|\mathbf{x}_0 - \mathbf{F}\mathbf{x}_0\|. \tag{3.8}$$

If we choose $\gamma := i/L[\sigma(b)-a]$ then we obtain a nice evaluation of the rate of convergence in (3.8), namely

$$\|\mathbf{F}^i\mathbf{x}_0 - \mathbf{x}\| \leq e_{\frac{i}{\sigma(b)-a}}(\sigma(b),a)\left(\frac{L[\sigma(b)-a]}{i}\right)^i \frac{i}{i-L[\sigma(b)-a]}\|\mathbf{x}_0 - \mathbf{F}\mathbf{x}_0\|.$$

We now present a simple example to illustrate Theorem 3.4.

Example 3.5 *Consider the scalar dynamic IVP*

$$\begin{aligned} x^\Delta &= 2\left[x^2+5\right]^{1/2} + t, \quad t \in [a,b]_{\mathbb{T}}; \\ x(a) &= A. \end{aligned}$$

We claim that this dynamic IVP has a unique solution for arbitrary $\mathbb{T}$.

Proof We will use Theorem 3.4. Consider

$$\begin{aligned} |f(t,p)-f(t,q)| &= |2\left[p^2+5\right]^{1/2} - 2\left[q^2+5\right]^{1/2}| \\ &\leq \sup_{r\in\mathbb{R}} |\frac{2r}{[r^2+5]^{1/2}}| \cdot |p-q|, \quad \text{by the mean value theorem} \\ &\leq 2|p-q| \end{aligned}$$

so that (3.4) holds with $L = 2$. The result now follows from Theorem 3.4. □

We now present a result on the dependency of solutions to the IVP (1.1), (1.3) with respect to initial values.

Theorem 3.6 *The solution furnished under the conditions of Theorem 3.4 is Lipschitz continuous in* $\mathbf{A}$*, uniformly in* t*. In fact, for any two initial conditions* $\mathbf{A}$*,* $\mathbf{B} \in \mathbb{R}^n$ *we have*

$$\|\mathbf{x}(t;\mathbf{A}) - \mathbf{x}(t;\mathbf{B})\| \leq e_L(t,a)\|\mathbf{A}-\mathbf{B}\|, \quad \forall t \in [a,\sigma(b)]_{\mathbb{T}}. \tag{3.9}$$

Proof Using (3.4) in a standard fashion we obtain the estimate

$$\|\mathbf{x}(t;\mathbf{A}) - \mathbf{x}(t;\mathbf{B})\| \leq L\int_a^t \|\mathbf{x}(s;\mathbf{A}) - \mathbf{x}(s;\mathbf{B})\|\,\Delta s + \|\mathbf{A}-\mathbf{B}\|, \quad t \in [a,\sigma(b)]_{\mathbb{T}}. \tag{3.10}$$

Now for $t \in [a,\sigma(b)]_{\mathbb{T}}$ let

$$E(t) := \int_a^t \|\mathbf{x}(s;\mathbf{A}) - \mathbf{x}(s;\mathbf{B})\|\,\Delta s$$

and so from (3.10) we have

$$E^\Delta - LE \leq \|\mathbf{A}-\mathbf{B}\|.$$

Dividing both sides in the previous line by an integrating factor $e_L(\sigma(t),a)$ we obtain

$$\left[\frac{E(t)}{e_L(t,a)}\right]^\Delta \leq \frac{\|\mathbf{A}-\mathbf{B}\|}{e_L(\sigma(t),a)}$$

with an integration from a to t on both sides yielding

$$E(t) \leq \frac{e_L(t,a) - 1}{L} \|\mathbf{A} - \mathbf{B}\|.$$

The result now follows via a substitution into (3.10). $\square$

Theorem 3.6 partially extends the ideas in [27, Theorem 2.6.1] from continuity of solutions with respect to initial conditions to Lipschitz continuity of solutions with respect to initial conditions. Furthermore, we do not impose the restriction $L[\sigma(b) - a] < 1$ in our ideas.

Our attention now turns to the dynamic IVP (1.2), (1.3). Although (1.2) appears to be similar to (1.1), there are genuine distinctions between the two in terms of mathematical theory, as the following theorem demonstrates.

Theorem 3.7 *Consider the dynamic IVP* (1.2), (1.3). *Let* $\mathbf{f} : [a,b]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ *be rd–continuous and let* $L > 0$ *be a constant. If*

$$\|\mathbf{f}(t,\mathbf{p}) - \mathbf{f}(t,\mathbf{q})\| \leq L\|\mathbf{p} - \mathbf{q}\|, \qquad \forall t \in [a,b]_{\mathbb{T}},\ (\mathbf{p},\mathbf{q}) \in \mathbb{R}^{2n}; \tag{3.11}$$

$$L \sup_{t \in [a,\sigma(b)]_{\mathbb{T}}} \mu(t) < 1; \tag{3.12}$$

then the dynamic IVP (1.2), (1.3) *has a unique solution. In addition, if a sequence of functions* $\{\mathbf{x}_i\}$ *is defined inductively by choosing any* $\mathbf{x}_0 \in C([a,\sigma(b)]_{\mathbb{T}};\mathbb{R}^n)$ *and setting*

$$\mathbf{x}_{i+1}(t) = \mathbf{A} + \int_a^t \mathbf{f}(s, \mathbf{x}_i^{\sigma}(s))\ \Delta s \tag{3.13}$$

then the sequence $\{\mathbf{x}_i\}$ *converges uniformly on* $[a,\sigma(b)]_{\mathbb{T}}$ *to the unique solution* $\mathbf{x}$ *of* (1.2), (1.3). *Furthermore,* $\mathbf{x}^{\Delta} \in C_{rd}([a,b]_{\mathbb{T}};\mathbb{R}^n)$.

Proof Since $\mathbf{f}$ is a rd–continuous function, (3.13) is well–defined. Let $L > 0$ be the constant defined in (3.11) and let $\beta := L\gamma$ where $\gamma > 1$ is a constant chosen such that $L|\mu|_0 = 1 - 1/\gamma$. Consider the complete metric space $(C([a,\sigma(b)]_{\mathbb{T}};\mathbb{R}^n), d_{\beta})$. Let

$$\mathbf{F} : C([a,\sigma(b)]_{\mathbb{T}};\mathbb{R}^n) \to C([a,\sigma(b)]_{\mathbb{T}};\mathbb{R}^n)$$

be defined by

$$[\mathbf{F}\mathbf{x}](t) := \int_a^t \mathbf{f}(s, \mathbf{x}^{\sigma}(s))\ \Delta s + \mathbf{A}, \quad t \in [a,\sigma(b)]_{\mathbb{T}}. \tag{3.14}$$

By Lemma 2.4, fixed–points of $\mathbf{F}$ will be solutions to the dynamic IVP (1.2), (1.3). Thus, we want to prove that there exists a unique $\mathbf{x}$ such that $\mathbf{F}\mathbf{x} = \mathbf{x}$. To do this, we show that $\mathbf{F}$ is a contractive map and Banach's fixed–point theorem will then apply. For any

$\mathbf{u}, \mathbf{v} \in C([a, \sigma(b)]_{\mathbb{T}}; \mathbb{R}^n)$, consider

$$\begin{aligned}
d_\beta(\mathbf{Fu}, \mathbf{Fv}) &\leq \sup_{t \in [a,\sigma(b)]_{\mathbb{T}}} \left[\frac{1}{e_\beta(t,a)} \int_a^t \|\mathbf{f}(s, \mathbf{u}^\sigma(s)) - \mathbf{f}(s, \mathbf{v}^\sigma(s))\| \, \Delta s \right] \\
&\leq \sup_{t \in [a,\sigma(b)]_{\mathbb{T}}} \left[\frac{1}{e_\beta(t,a)} \int_a^t L\|\mathbf{u}^\sigma(s) - \mathbf{v}^\sigma(s)\| \, \Delta s \right], \quad \text{from (3.11)} \\
&= L \sup_{t \in [a,\sigma(b)]_{\mathbb{T}}} \left[\frac{1}{e_\beta(t,a)} \int_a^t e_\beta(\sigma(s), a) \frac{\|\mathbf{u}^\sigma(s) - \mathbf{v}^\sigma(s)\|}{e_\beta(\sigma(s), a)} \, \Delta s \right] \\
&\leq L d_\beta(\mathbf{u}, \mathbf{v}) \sup_{t \in [a,\sigma(b)]_{\mathbb{T}}} \left[\frac{1}{e_\beta(t,a)} \int_a^t e_\beta(\sigma(s), a) \, \Delta s \right] \\
&= L d_\beta(\mathbf{u}, \mathbf{v}) \sup_{t \in [a,\sigma(b)]_{\mathbb{T}}} \left[\frac{1}{e_\beta(t,a)} \int_a^t (1 + \beta\mu(s)) e_\beta(s, a) \, \Delta s \right] \\
&\leq L(1 + \beta|\mu|_0) d_\beta(\mathbf{u}, \mathbf{v}) \sup_{t \in [a,\sigma(b)]_{\mathbb{T}}} \left[\frac{1}{e_\beta(t,a)} \int_a^t e_\beta(s, a) \, \Delta s \right] \\
&= L(1 + \beta|\mu|_0) d_\beta(\mathbf{u}, \mathbf{v}) \sup_{t \in [a,\sigma(b)]_{\mathbb{T}}} \left[\frac{1}{e_\beta(t,a)} \left(\frac{e_\beta(t,a) - 1}{\beta} \right) \right] \\
&= \left(\frac{1}{\gamma} + L|\mu|_0 \right) d_\beta(\mathbf{u}, \mathbf{v}) \sup_{t \in [a,\sigma(b)]_{\mathbb{T}}} \left[1 - \frac{1}{e_\beta(t,a)} \right], \quad \text{since } \beta = L\gamma, \\
&= d_\beta(\mathbf{u}, \mathbf{v}) \left[1 - \frac{1}{e_\beta(\sigma(b), a)} \right] < d_\beta(\mathbf{u}, \mathbf{v}).
\end{aligned}$$

We see that $\mathbf{F}$ is a contractive map and Banach's fixed–point theorem applies, yielding the existence of a unique fixed–point $\mathbf{x}$ of $\mathbf{F}$. In addition, from Banach's theorem, the sequence $\{\mathbf{x}_i\}$ defined in (3.13) converges uniformly in the TZ–norm $\|\cdot\|_\beta$ and thus the sequence $\{\mathbf{x}_i\}$ converges uniformly in the sup–norm $\|\cdot\|_0$ to that fixed–point $\mathbf{x}$. This completes the proof. □

Theorem 3.7 gives existence and uniqueness of solutions to (1.2), (1.3) for those time scales where the points are not spaced "too far apart", whereas Theorem 3.4 does not involve any such restriction. Notice that condition (3.12) in Theorem 3.7 is less restrictive than the "usual" assumption $L[\sigma(b) - a] < 1$.

It is a fact of life that not all maps are contractions in the global sense. However, many maps are *locally* contractive, that is, they are contractive within certain balls. We now investigate this avenue and the following local version of the Banach fixed–point theorem will be useful for our study.

Corollary 3.8 (local Banach, [12] pp.10–11) *Let (Y, d) be a complete metric space and let $B_r(y_0)$ represent an open ball in Y with centre y_0 and radius r. Let $F : B_r \to Y$ be contractive with contraction constant $\alpha < 1$. If $d(F(y_0), y_0) < (1 - \alpha)r$ then F has a fixed–point.*

The following result gives conditions that guarantee local existence and uniqueness of solutions to our dynamic equations, with the solutions lying in certain balls.

Theorem 3.9 *Consider the dynamic IVP* (1.1), (1.3). *Let* $\mathbf{f} : [a,b]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ *be rd–continuous with* M *and* L *being positive constants. If*

$$\|\mathbf{f}(t,\mathbf{p}) - \mathbf{f}(t,\mathbf{q})\| \le L\|\mathbf{p}-\mathbf{q}\|, \qquad \forall t \in [a,b]_{\mathbb{T}},\ \|\mathbf{p}-\mathbf{A}\| \le M,\ \|\mathbf{q}-\mathbf{A}\| \le M; \tag{3.15}$$

$$\int_a^{\sigma(b)} \|\mathbf{f}(s,\mathbf{A})\|\,\Delta s \ < \ \frac{M}{[e_L(\sigma(b),a)]^2}; \tag{3.16}$$

then the dynamic IVP (1.1), (1.3) *has at least one solution, with a unique solution satisfying* $d_L(\mathbf{x},\mathbf{A}) < M/e_L(\sigma(b),a)$.

Proof Choose $R > 0$ such that $Re_L(\sigma(b),a) = M$ and consider the complete metric space $(C([a,\sigma(b)]_{\mathbb{T}};\mathbb{R}^n), d_L)$. Now consider the open ball $B_R(\mathbf{A}) \subset C([a,\sigma(b)]_{\mathbb{T}};\mathbb{R}^n)$ defined by

$$B_R(\mathbf{A}) := \{\mathbf{x} \in C([a,\sigma(b)]_{\mathbb{T}};\mathbb{R}^n) \ : \ d_L(\mathbf{x},\mathbf{A}) < R\}.$$

together the operator $\mathbf{F}$ defined in (3.6) with

$$\mathbf{F} : B_R(\mathbf{A}) \to C([a,\sigma(b)]_{\mathbb{T}};\mathbb{R}^n).$$

We show that the conditions of the local Banach corollary are satisfied. In a similar fashion to the proof of Theorem 3.4 we obtain for all $\mathbf{u}$, $\mathbf{v} \in B_R(\mathbf{A})$

$$d_L(\mathbf{Fu},\mathbf{Fv}) \le \left[1 - \frac{1}{e_L(\sigma(b),a)}\right] d_L(\mathbf{u},\mathbf{v}).$$

So we see that $\mathbf{F}$ is a contractive map with contraction constant $\alpha = 1 - 1/e_L(\sigma(b),a)$.

Now consider

$$\begin{aligned} d_L(\mathbf{F}(\mathbf{A}),\mathbf{A}) &= \sup_{t\in[a,\sigma(b)]_{\mathbb{T}}} \left[\frac{1}{e_L(t,a)} \int_a^{\sigma(b)} \|\mathbf{f}(s,\mathbf{A})\|\,\Delta s\right] \\ &\le \int_a^{\sigma(b)} \|\mathbf{f}(s,\mathbf{A})\|\,\Delta s \\ &< \frac{M}{[e_L(\sigma(b),a)]^2}, \quad \text{by (3.16)} \\ &= \frac{R}{e_L(\sigma(b),a)} \\ &= (1-\alpha)R \end{aligned}$$

so that all of the conditions of the local Banach corollary are satisfied. Thus $\mathbf{F}$ has a unique fixed–point $\mathbf{x} \in B_R(\mathbf{A})$ so the dynamic IVP also has a unique solution in $B_R(\mathbf{A})$. □

We now present a simple example to illustrate Theorem 3.9.

Example 3.10 *Consider the scalar dynamic IVP*

$$\begin{aligned} x^{\Delta} &= x^2 + t + \sigma(t), \quad t \in [a,b]_{\mathbb{T}}; \\ x(a) &= 0. \end{aligned}$$

We claim that this dynamic IVP has a solution when $\sigma(b) - a$ *is sufficiently small.*

Proof We will use Theorem 3.9. Let $M > 0$ is an arbitrary constant. For $|p| \leq M$ and $|q| \leq M$ consider

$$\begin{aligned} |f(t,p) - f(t,q)| &\leq |p+q|\ |p-q| \\ &\leq 2M|p-q| \end{aligned}$$

so that (3.15) holds with $L = 2M$. Now consider

$$\begin{aligned} \int_a^{\sigma(b)} \|\mathbf{f}(s,\mathbf{A})\|\ \Delta s &= \int_a^{\sigma(b)} (s + \sigma(s))\ \Delta s \\ &= [\sigma(b)]^2 - a^2 \end{aligned}$$

which can be made smaller than $M/[e_L(\sigma(b),a)]^2$ provided $\sigma(b) - a$ is sufficiently small and thus (3.16) holds.

The result now follows from Theorem 3.9. $\square$

Note that the problem in the previous example does not satisfy (3.4) in the global sense and thus Theorem 3.4 does not apply.

4 Topological degree approach

In this section we formulate some new results that guarantee the existence of at least one solution to the dynamic IVPs: (1.1), (1.3); and (1.2), (1.3). Fixed–point theorems based on the ideas of topological degree [29, Chap.4] will be the main tools to be used in conjunction with a new definition of measuring distance via the TZ–norm in normed spaces. In particular, Schäfer's fixed–point theorem will be employed, rather than Banach's fixed–point theorem.

A convenient advantage of Schäfer's theorem is that no explicit knowledge of topological degree theory is needed to verify the conditions of the theorem. On one hand, Schäfer's theorem is very wide–ranging, but on the other hand, it does not provide as much information as Banach's theorem. Thus, the focus in this section is on obtaining qualitative information about solutions, in particular, existence of at least one solution, under mild assumptions on $\mathbf{f}$.

Theorem 4.1 (Schäfer, [29] Theorem 4.4.12) *Let X be a normed space with $H : X \to X$ a compact mapping. If the set*

$$S := \{u \in X : \ u = \lambda H u \quad \textit{for some } \lambda \in [0,1)\}$$

is bounded then H has at least one fixed–point.

Recall that a mapping between normed spaces is compact if it is continuous and carries bounded sets into relatively compact sets.

Lemma 4.2 *Consider the normed space $(C([a,\sigma(b)]_{\mathbb{T}};\mathbb{R}^n), \|\cdot\|_0)$ and consider the map $\mathbf{F} : C([a,\sigma(b)]_{\mathbb{T}};\mathbb{R}^n) \to C([a,\sigma(b)]_{\mathbb{T}};\mathbb{R}^n)$ defined in* (3.6)*. If $\mathbf{f} : [a,b]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous then $\mathbf{F}$ is a compact map.*

Proof Our argument follows that of [12, pp.52–53] and so is only sketched. We show that the conditions of the Arzela–Ascoli theorem are satisfied. That is, given $\{\mathbf{x}_i\}$ with $\|\mathbf{x}_i\|_0 \leq r$ we show that the sequence $\mathbf{v}_i := \mathbf{F}(\mathbf{x}_i)$ is bounded and equicontinuous.

(a): We claim that $\{\mathbf{v}_i\}$ is bounded. Let

$$M := \sup\{\|\mathbf{f}(t,\mathbf{p})\| : t \in [a,\sigma(b)]_{\mathbb{T}},\ \|\mathbf{p}\| \leq r\} < \infty.$$

We have

$$\begin{aligned}
\|\mathbf{v}_i\|_0 &:= \sup_{t\in[a,\sigma(b)]_{\mathbb{T}}} \|\mathbf{v}_i(t)\| && (4.1)\\
&\leq \sup_{t\in[a,\sigma(b)]_{\mathbb{T}}} \int_a^t \|\mathbf{f}(s,\mathbf{x}_i(s))\|\ \Delta s + \|\mathbf{A}\| && (4.2)\\
&\leq [\sigma(b)-a]M + \|\mathbf{A}\| && (4.3)
\end{aligned}$$

and hence $\{\mathbf{v}_i\}$ is bounded.

(b): We claim that $\{\mathbf{v}_i\}$ is equicontinuous. Let $\varepsilon > 0$ be given and for $t_1,\ t_2 \in [a,\sigma(b)]_{\mathbb{T}}$ consider

$$\begin{aligned}
\|\mathbf{v}_i(t_1) - \mathbf{v}_i(t_2)\| &\leq \int_{\min\{t_1,t_2\}}^{\max\{t_1,t_2\}} \|\mathbf{f}(s,\mathbf{x}_i(s)\|\ \Delta s\\
&\leq M|t_1-t_2|\\
&< \varepsilon, \quad \text{whenever } |t_1-t_2| < \delta(\varepsilon) := \varepsilon/M
\end{aligned}$$

and hence $\{\mathbf{v}_i\}$ is equicontinuous.

The result now follows from the Arzela–Ascoli Theorem [32, p.3]. □

The following existence theorem permits linear growth of $\|\mathbf{f}(t,\mathbf{p})\|$ in $\|\mathbf{p}\|$.

Theorem 4.3 *Consider the dynamic IVP* (1.1), (1.3). *Let* $\mathbf{f} : [a,b]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ *be continuous and* $L > 0$, $N \geq 0$ *be constants. If*

$$\|\mathbf{f}(t,\mathbf{p})\| \leq L\|\mathbf{p}\| + N, \quad \forall t \in [a,b]_{\mathbb{T}},\ \mathbf{p} \in \mathbb{R}^n; \tag{4.4}$$

then the dynamic IVP (1.1), (1.3) *has at least one solution.*

Proof We will use Schäfer's theorem. Let $L > 0$ be the constant defined in (4.4). Consider the normed space

$$(C([a,\sigma(b)]_{\mathbb{T}};\mathbb{R}^n), \|\cdot\|_L)$$

with the family of equations

$$\mathbf{x} = \lambda\mathbf{F}\mathbf{x}, \quad \lambda \in [0,1); \tag{4.5}$$

where

$$\mathbf{F} : C([a,\sigma(b)]_{\mathbb{T}};\mathbb{R}^n) \to C([a,\sigma(b)]_{\mathbb{T}};\mathbb{R}^n)$$

is defined in (3.6). Note that $\mathbf{F}$ is a compact map from Lemma 4.2. From Lemma 2.3, fixed points of $\mathbf{F}$ will be solutions to the dynamic IVP (1.1), (1.3).

For a fixed $\lambda \in [0,1)$ let $\mathbf{x} \in C([a,\sigma(b)]_{\mathbb{T}};\mathbb{R}^n)$ be a solution to (4.5). We then have

$$
\begin{aligned}
\|\mathbf{x}\|_L &\leq \sup_{t\in[a,\sigma(b)]_{\mathbb{T}}} \left[\frac{1}{e_L(t,a)}\left(\int_a^t \|\lambda\mathbf{f}(s,\mathbf{x}(s))\|\ \Delta s + \|\lambda\mathbf{A}\|\right)\right] \\
&\leq \sup_{t\in[a,\sigma(b)]_{\mathbb{T}}} \left[\frac{1}{e_L(t,a)}\left(\int_a^t (L\|\mathbf{x}(s)\| + N)\ \Delta s + \|\mathbf{A}\|\right)\right], \quad \text{from (4.4)} \\
&\leq \sup_{t\in[a,\sigma(b)]_{\mathbb{T}}} \left[\frac{1}{e_L(t,a)}\left(\int_a^t e_L(s,a)\frac{L\|\mathbf{x}(s)\|}{e_L(s,a)}\ \Delta s + N[t-a] + \|\mathbf{A}\|\right)\right] \\
&\leq L\|\mathbf{x}\|_L \sup_{t\in[a,\sigma(b)]_{\mathbb{T}}} \left[\frac{1}{e_L(t,a)}\int_a^t e_L(s,a)\ \Delta s\right] + N[\sigma(b)-a] + \|\mathbf{A}\| \\
&= \|\mathbf{x}\|_L\left[1 - \frac{1}{e_L(\sigma(b),a)}\right] + N[\sigma(b)-a] + \|\mathbf{A}\|
\end{aligned}
$$

and a rearrangement leads to

$$\|\mathbf{x}\|_L \leq e_L(\sigma(b),a)(N[\sigma(b)-a] + \|\mathbf{A}\|).$$

Thus, the set of possible solutions to the family (4.5) is bounded *a priori*, with the bound being independent of λ. Schäfer's theorem now applies to $\mathbf{F}$, yielding the existence of at least one fixed–point. Hence the dynamic IVP (1.1), (1.3) also has at least one solution. □

If we had used the sup–norm $\|\cdot\|_0$ in the proof of Theorem 4.3 rather than the TZ–norm $\|\cdot\|_L$ then we would have needed an additional assumption in Theorem 4.3, namely, $L[\sigma(b)-a)] < 1$.

As an application of Theorem 4.3 we present the following example.

Example 4.4 *Consider the dynamic IVP*

$$
\begin{aligned}
x_1^{\Delta} &= x_1 + x_2\cos(tx_2). \quad t \in [a,b]_{\mathbb{T}}; \\
x_2^{\Delta} &= x_2 - x_1\cos(tx_2);
\end{aligned}
$$

subject to any initial condition $\mathbf{x}(a) = \mathbf{A} \neq \mathbf{0}$. *We claim that this dynamic IVP has at least one solution for arbitrary* $\mathbb{T}$.

Proof We will use Theorem 4.3. Let $\mathbf{p} := (p_1,p_2)$. Then

$$
\begin{aligned}
\|\mathbf{f}(t,\mathbf{p})\| &= \left[p_1^2 + p_2^2\cos^2(tp_2) + p_2^2 + p_1^2\cos^2(tp_2)\right]^{1/2} \\
&\leq \left[2(p_1^2+p_2^2)\right]^{1/2} = \sqrt{2}\|\mathbf{p}\|
\end{aligned}
$$

so that (4.4) holds with $L = \sqrt{2}$ and $N = 0$. The result now follows from Theorem 4.3. □

The following is an important special case of Theorem 4.3.

Theorem 4.5 *Consider the dynamic IVP* (1.1), (1.3) *with* $\mathbf{f} : [a,b]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ *continuous. If there exists a non–negative constant* N *such that*

$$\|\mathbf{f}(t,\mathbf{p})\| \leq N, \quad \forall t \in [a,b]_{\mathbb{T}},\ \mathbf{p} \in \mathbb{R}^n; \tag{4.6}$$

then the dynamic IVP (1.1), (1.3) *has at least one solution.*

Proof It is easy to see that if (4.6) holds then (4.4) holds and the result now follows from Theorem 4.3. □

In a similar fashion to Theorem 4.5 we have the following result for the dynamic IVP (1.2), (1.3).

Theorem 4.6 *Consider the dynamic IVP* (1.2), (1.3) *with* $\mathbf{f} : [a, b]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ *continuous. If there exists a non–negative constant* N *such that* (4.6) *holds then the dynamic IVP* (1.2), (1.3) *has at least one solution.*

Proof The idea of the proof follows that of Theorem 4.5 and so is omitted. □

The following two results allow superlinear growth of $\|\mathbf{f}(t, \mathbf{p})\|$ in $\|\mathbf{p}\|$ and, because of the nature of their proofs, appear to be uniquely applicable to the dynamic IVP (1.2), (1.3) rather than to (1.1), (1.3).

Theorem 4.7 *Consider the dynamic IVP* (1.2), (1.3) *with* $\mathbf{f} : [a, b]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ *continuous. If there exist non–negative constants* L *and* N *such that*

$$\|\mathbf{f}(t, \mathbf{p})\| \leq -2L\langle \mathbf{p}, f(t, \mathbf{p})\rangle + N, \quad \forall t \in [a, b]_{\mathbb{T}},\ \mathbf{p} \in \mathbb{R}^n; \tag{4.7}$$

then the dynamic IVP (1.2), (1.3) *has at least one solution.*

Proof We will use Schäfer's theorem. Consider the normed space

$$(C([a, \sigma(b)]_{\mathbb{T}}; \mathbb{R}^n), \|\cdot\|_0)$$

and the family of equations

$$\mathbf{x} = \lambda \mathbf{F}\mathbf{x}, \quad \lambda \in [0, 1); \tag{4.8}$$

where

$$\mathbf{F} : C([a, \sigma(b)]_{\mathbb{T}}; \mathbb{R}^n) \to C([a, \sigma(b)]_{\mathbb{T}}; \mathbb{R}^n)$$

is defined in (3.6). Note that $\mathbf{F}$ is a compact map by a similar argument as in the proof of Lemma 4.2.

For a fixed $\lambda \in [0, 1)$ let $\mathbf{x} \in C([a, \sigma(b)]_{\mathbb{T}}; \mathbb{R}^n)$ be a solution to (4.8). Then, for a fixed $\lambda \in [0, 1)$, $\mathbf{x}^{\Delta} \in C_{rd}([a, b]_{\mathbb{T}}; \mathbb{R}^n)$ and $\mathbf{x}$ is also a solution to

$$\begin{aligned} \mathbf{x}^{\Delta} &= \lambda \mathbf{f}(t, \mathbf{x}^{\sigma}), \quad t \in [a, b]_{\mathbb{T}}; && (4.9)\\ \mathbf{x}(a) &= \lambda \mathbf{A}. && (4.10) \end{aligned}$$

If $r(t) := \|\mathbf{x}(t)\|^2$ for all $t \in [a, \sigma(b)]_{\mathbb{T}}$ then for each $t \in [a, b]_{\mathbb{T}}$ we have, from the product rule [5, Theorem 1.20 (iii)],

$$\begin{aligned} r^{\Delta}(t) &= \langle \mathbf{x}(t) + \mathbf{x}^{\sigma}(t), \mathbf{x}^{\Delta}(t)\rangle \\ &= 2\langle \mathbf{x}^{\sigma}(t), \mathbf{x}^{\Delta}(t)\rangle - \mu(t)\|\mathbf{x}^{\Delta}(t)\|^2, \quad \text{by Theorem 1.6, (iv).} \end{aligned}$$

In addition, if (4.7) holds then for $\lambda \in [0, 1)$

$$\|\lambda \mathbf{f}(t, \mathbf{p})\| \leq -2L\langle \mathbf{p}, \lambda \mathbf{f}(t, \mathbf{p})\rangle + N, \quad \forall t \in [a, b]_{\mathbb{T}},\ \mathbf{p} \in \mathbb{R}^n.$$

Taking norms in (4.8) we have

$$
\begin{aligned}
& \|\mathbf{x}\|_0 = \|\lambda \mathbf{F}\mathbf{x}\|_0 \\
\leq & \sup_{t\in[a,\sigma(b)]_{\mathbb{T}}} \int_a^t \|\lambda \mathbf{f}(s,\mathbf{x}^\sigma(s))\| \ \Delta s + \|\lambda \mathbf{A}\| \\
\leq & \sup_{t\in[a,\sigma(b)]_{\mathbb{T}}} \int_a^t (-2L\langle \mathbf{x}^\sigma(s), \lambda \mathbf{f}(s,\mathbf{x}^\sigma(s))\rangle + N) \ \Delta s + \|\mathbf{A}\| \\
\leq & \sup_{t\in[a,\sigma(b)]_{\mathbb{T}}} \int_a^t \big(-2L\langle \mathbf{x}^\sigma(s), \lambda \mathbf{f}(s,\mathbf{x}^\sigma(s))\rangle + L\mu(s)\|\lambda \mathbf{f}(s,\mathbf{x}^\sigma)\|^2 + N\big) \ \Delta s + \|\mathbf{A}\| \\
= & \sup_{t\in[a,\sigma(b)]_{\mathbb{T}}} \int_a^t \big(-Lr^\Delta(s) + N\big) \ \Delta s + \|\mathbf{A}\| \\
= & \sup_{t\in[a,\sigma(b)]_{\mathbb{T}}} (L[r(a)-r(t)] + N(t-a)) + \|\mathbf{A}\| \\
\leq & L\|\mathbf{A}\|^2 + N[\sigma(b)-a] + \|\mathbf{A}\|.
\end{aligned}
$$

Thus, the set of possible solutions to the family (4.8) is bounded *a priori*, with the bound being independent of λ. Schäfer's theorem now applies to $\mathbf{F}$, yielding the existence of at least one fixed–point. Hence the dynamic IVP (1.2), (1.3) also has at least one solution. □

The following result compliments Theorem 4.7 as it also permits superlinear growth in $\mathbf{f}$.

Theorem 4.8 *Consider the dynamic IVP* (1.2), (1.3) *with* $\mathbf{f} : [a,b]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ *continuous. If there exists a non–negative constant* K *such that*

$$\langle \mathbf{p}, f(t,\mathbf{p})\rangle \leq K, \quad \forall t \in [a,b]_{\mathbb{T}}, \ \mathbf{p} \in \mathbb{R}^n; \tag{4.11}$$

then the dynamic IVP (1.2), (1.3) *has at least one solution.*

Proof We will use Schäfer's theorem. Consider: the normed space

$$(C([a,\sigma(b)]_{\mathbb{T}};\mathbb{R}^n), \|\cdot\|_0);$$

the family of equations defined in (4.8); and the family of dynamic IVPs (4.9), (4.10). Let the function r be defined as in the proof of Theorem 4.7 where $\mathbf{x}$ is a solution to (4.9), (4.10) for a fixed $\lambda \in [0,1]$. We then have, for each $t \in [a,b]_{\mathbb{T}}$,

$$
\begin{aligned}
r^\Delta(t) &= 2\langle \mathbf{x}^\sigma(t), \mathbf{x}^\Delta(t)\rangle - \mu(t)\|\mathbf{x}^\Delta(t)\|^2 \\
&\leq 2\langle \mathbf{x}^\sigma(t), \lambda \mathbf{f}(t,\mathbf{x}^\sigma(t))\rangle \\
&\leq \lambda K \leq K.
\end{aligned}
$$

Hence

$$\int_a^t r^\Delta(s) \ \Delta s \leq K \int_a^t \Delta s, \quad t \in [a,\sigma(b)]_{\mathbb{T}};$$

so that

$$\|\mathbf{x}\|_0^2 \leq \|\mathbf{A}\|^2 + K(\sigma(b)-a).$$

Thus, the set of possible solutions to the family (4.8) is bounded *a priori*, with the bound being independent of λ. Note that $\mathbf{F}$ is a compact map. Schäfer's theorem now applies to $\mathbf{F}$, yielding the existence of at least one fixed–point. Hence the dynamic IVP (1.2), (1.3) also has at least one solution. $\square$

The idea for Theorem 4.8 was motivated by the papers [11] and [18].

Comparing Theorems 4.7 and 4.8 we see that if (4.7) holds then (4.11) holds so that Theorem 4.7 is a special case of Theorem 4.8.

We now demonstrate the applicability of Theorem 4.8 via an example.

Example 4.9 *Consider the dynamic IVP*

$$\begin{aligned} x_1^{\Delta} &= -x_1^3 t - x_2 + 1. \quad t \in [a,b]_{\mathbb{T}}; \\ x_2^{\Delta} &= -x_2^3 t + x_1 + 1; \end{aligned}$$

subject to any initial condition $\mathbf{x}(a) = \mathbf{A} \neq \mathbf{0}$. *We claim that this dynamic IVP has at least one solution for arbitrary* $\mathbb{T}$ *with* $a \geq 1$.

Proof We will use Theorem 4.8. Let $\mathbf{p} := (p_1, p_2)$. Then

$$\begin{aligned} \langle \mathbf{p}, \mathbf{f}(t,\mathbf{p}) \rangle &= -p_1^4 t + p_1 - p_2^4 t + p_2 \\ &\leq -p_1^4 + p_1 - p_2^4 + p_2 \leq 2 \end{aligned}$$

so that (4.11) holds with $K = 2$. The result now follows from Theorem 4.8. $\square$

5 Higher–order equations

The ideas of Sections 3 and 4 are now extended to treat higher–order dynamic IVPs of the type

$$x^{\Delta^n} = f(t, x, x^{\Delta}, x^{\Delta^2}, \ldots, x^{\Delta^{n-1}}), \quad t \in [a,b]_{\mathbb{T}}; \tag{5.1}$$

$$x(a) = A_1; \quad x^{\Delta}(a) = A_2; \quad \cdots; \quad x^{\Delta^{n-1}}(a) = A_{n-1}; \tag{5.2}$$

where: $x^{\Delta^i} := [x^{\Delta^{i-1}}]^{\Delta}$; $f : [a,b]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}$; and each A_i is a given constant in $\mathbb{R}$.

We define $\sigma^i(b)$ as: $\sigma^1(b) := \sigma(b)$; and $\sigma^{i+1}(b) := \sigma^i(\sigma(b))$. A solution to (5.1), (5.2) is a function $x : [a, \sigma^n(b)]_{\mathbb{T}} \to \mathbb{R}$ that is n-times delta–differentiable and satisfies (5.1), (5.2).

The function $f : [a,b]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}$ in (5.1) is said to be rd–continuous if : f is continuous at each $(t, x_1, \ldots, x_n)$ where t is right–dense; and the limits

$$\lim_{(s,y_1,\ldots,y_n)\to(t^-,x_1,\ldots,x_n)} f(s, y_1, \ldots, y_n) \quad \text{and} \quad \lim_{(y_1,\ldots,y_n)\to(x_1,\ldots,x_n)} f(t, y_1, \ldots, y_n)$$

both exist and are finite at each $(t, x_1, \ldots, x_n)$ where t is left–dense.

We now present our first result for the higher–order IVP (5.1), (5.2).

Theorem 5.1 *Consider the dynamic, higher–order IVP* (5.1), (5.2). *Let* $f : [a,b]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}$ *be rd–continuous and let* L_1 *be a positive constant. If*

$$[f(t,p_1,\ldots,p_n) - f(t,q_1,\ldots,q_n)]^2 \leq L_1[(p_1-q_1)^2 + \ldots + (p_n-q_n)^2], \tag{5.3}$$
$$\forall t \in [a,b]_{\mathbb{T}}, \quad \text{and each } p_i, q_i \in \mathbb{R};$$

then the dynamic, higher–order IVP (5.1), (5.2) *has a unique solution.*

Proof The idea of the proof involves a suitable substitution and the use of Banach's theorem. Introduce new variables $x_1, \ldots, x_n$ with:

$$x_1 := x; \quad x_2 := x^{\Delta}; \quad x_3 := x^{\Delta^2}; \quad \cdots \quad ; \quad x_{n-1} := x^{\Delta^{n-2}}; \quad x_n := x^{\Delta^{n-1}}.$$

If we take the delta derivative of each x_i above then we obtain a system of n, first–order dynamic equations

$$x_1^{\Delta} = x_2; \quad x_2^{\Delta} = x_3; \quad \cdots \quad ; \quad x_{n-1}^{\Delta} = x_n; \quad x_n^{\Delta} = f(t, x_1, \ldots, x_n);$$

so that our higher–order problem may now be written as

$$\begin{aligned} \mathbf{x}^{\Delta} &= \mathbf{f}(t, \mathbf{x}), \quad t \in [a,b]_{\mathbb{T}}; && (5.4) \\ \mathbf{x}(a) &= \mathbf{A} && (5.5) \end{aligned}$$

for suitable $\mathbf{f}$ and $\mathbf{A}$. Now for any $\mathbf{p}$, $\mathbf{q} \in \mathbb{R}^n$ consider

$$\begin{aligned} & \|\mathbf{f}(t,\mathbf{p}) - \mathbf{f}(t,\mathbf{q})\| \\ = \; & \left[(p_2 - q_2)^2 + (p_3 - q_3)^2 + \ldots + (f(t, p_1, \ldots, p_n) - f(t, q_1, \ldots, q_n))^2\right]^{1/2} \\ \leq \; & \left[(p_1 - q_1)^2 + (p_2 - q_2)^2 + \ldots + L_1((p_1 - q_1)^2 + \ldots + (p_n - q_n)^2)\right]^{1/2} \\ = \; & (1 + L_1)^{1/2} \|\mathbf{p} - \mathbf{q}\| \end{aligned}$$

so that (3.4) holds with $L := (1 + L_1)^{1/2}$. Thus the dynamic IVP (5.4), (5.5) has a unique solution from Theorem 3.4. The existence of a solution to the higher–order IVP (5.1), (5.2) now follows. □

The following result allows linear–type growth of $|f|$ in its variables.

Theorem 5.2 *Consider the dynamic, higher–order IVP* (5.1), (5.2)*. Let* $f : [a,b]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}$ *be continuous and* $L_1 > 0$, $N_1 \geq 0$ *be constants. If*

$$|f(t, p_1, \ldots, p_n)| \leq L_1[p_1^2 + \ldots + p_n^2]^{1/2} + N_1, \quad \forall t \in [a,b]_{\mathbb{T}}, \ (p_1 \ldots, p_n) \in \mathbb{R}^n; \qquad (5.6)$$

then the dynamic, higher–order IVP (5.1), (5.2) *has at least one solution.*

Proof The idea of the proof involves a suitable substitution and the use of Schäfer's theorem. Introduce new variables as in the proof of Theorem 5.1 and consider the IVP (5.4), (5.5). Note that $\mathbf{F}$ is a compact map from Lemma 4.2.. Now for any $(t, \mathbf{p}) \in [a,b]_{\mathbb{T}} \times \mathbb{R}^n$ consider

$$\begin{aligned} \|\mathbf{f}(t,\mathbf{p})\| &= \left[p_2^2 + p_3^2 + \ldots + (f(t, p_1, \ldots, p_n))^2\right]^{1/2} \\ &\leq \left[p_1^2 + p_2^2 + \ldots + p_{n-1}^2\right]^{1/2} + |f(t, p_1, \ldots, p_n)| \\ &\leq (1 + L_1)\left[p_1^2 + p_2^2 + \ldots + p_n^2\right]^{1/2} + N_1 \\ &= (1 + L_1)\|\mathbf{p}\| + N_1 \end{aligned}$$

so that (4.4) holds with $L := (1 + L_1)$ and $N := N_1$. Thus the dynamic IVP (5.4), (5.5) has at least one solution from Theorem 4.3. The existence of a solution to the higher–order IVP (5.1), (5.2) now follows. □

6 A simple example from economics

In this section we formulate and analyse a simple model from economics in the time scale setting, known as *a Keynesian–Cross model with "lagged" income.*

In a simple closed economy, the dynamics of: aggregate demand, D; aggregate income, y; aggregate consumption, C; aggregate investment, I; government spending, G; are given by three simple equations, namely:

$$\begin{aligned} D(t) &= C(t) + I + G; && (6.1)\\ C(t) &= C_0 + cy(t); && (6.2)\\ y^{\Delta} &= \delta[D^{\sigma} - y], \quad t \geq a; && (6.3) \end{aligned}$$

where $\delta < 1$ is a positive constant known as the "speed of adjustment term" and C_0, c are non–negative constants. To keep the model very simple, G and I are assumed to be constant in (6.1), and current consumption is assumed to depend on current income in (6.2). Equation (6.3) means that the change in income is a fraction of excess demand at $\sigma(t)$ over income at t.

The above example is a generalisation of the classical Keynesian–Cross model involving difference equations given in [16, p.23] for the special case $\mathbb{T} = \mathbb{Z}$.

If we substitute (6.1) and (6.2) into (6.3) then we obtain

$$\begin{aligned} y^{\Delta} &= \delta[C_0 + cy^{\sigma} + I + G - y]\\ &:= h(t, y, y^{\sigma}). \end{aligned}$$

The above equation can be recast into the form (1.1) using the simple, useful formula $y^{\sigma} = y + \mu y^{\Delta}$ and if $1 - \delta c\mu(t) \neq 0$ for $t \geq a$ then a substitution and rearrangement leads to

$$\begin{aligned} y^{\Delta} &= \frac{\delta(c-1)}{1 - \delta c\mu(t)} y + \frac{\delta(c_0 + I + G)}{1 - \delta c\mu(t)}\\ &:= f(t)y + g(t). && (6.4) \end{aligned}$$

It is easy to verify that the dynamic equation in (6.4) satisfies the conditions of Theorem 3.4 and so a unique solution y to our problem exists on any compact interval of the type $[a, \sigma(b)]_{\mathbb{T}}$. However, we can go further and define the solution for all $t \geq a$ by solving the linear dynamic equation (6.4) directly. Using the techniques in [5, Chap.2] we obtain

$$y(t) = e_f(t, a) \left[y(a) + \int_a^t \frac{g(s)}{e_f(\sigma(s), a)} \Delta s \right], \quad t \geq a.$$

7 Separation of variables approach

Bohner and Peterson [5, p.51] define a dynamic equation to be of first–order if it is of the form

$$x^{\Delta} = f(t, x, x^{\sigma}). \tag{7.1}$$

Although the presence of *both* x and x^{σ} in (7.1) may appear to be somewhat strange, their appearance can naturally occur, as the example from economics presented in Section 6 clearly illustrates.

We will look at a special case of (7.1) and develop a method to find its solution. The method is akin to the separation of variables approach for first–order ordinary differential equations. The ideas will rely heavily on the chain rule for time scales.

Theorem 7.1 (Chain rule for $\mathbb{T}$) *Let $V : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose that $x : \mathbb{T} \to \mathbb{R}$ is delta–differentiable. Then $V \circ x$ is delta–differentiable and*

$$[V(x(t))]^{\Delta} = \left(\int_0^1 V'(x(t) + h\mu(t)x^{\Delta}(t)) \, dh \right) \cdot x^{\Delta}(t).$$

Proof See Pötzsche [35] or Bohner and Peterson [5, Theorem 1.90]. □

We will call the dynamic equation (7.1) "separable" if is has the form

$$x^{\Delta} = f(t, x, x^{\sigma}) = \frac{j(t)}{k(x, x^{\sigma})} \tag{7.2}$$

for a right–dense continuous function j, and a continuous function k.

Theorem 7.2 (Separation of variables) *Consider the separable dynamic equation* (7.2)*. If there exists a function $K \in C^1(\mathbb{R}; \mathbb{R})$ such that*

$$\int_0^1 K'(x + h[x^{\sigma} - x]) \, dh = k(x, x^{\sigma}) \tag{7.3}$$

then the solution to (7.2)*, subject to the initial condition $x(a) = A$, is given implicitly by*

$$K(x(t)) = \int_a^t j(s) \, \Delta s + K(A).$$

If, in addition, K is globally one–to–one then the solution x can be explicitly obtained.

Proof If we "separate" the variables in (7.2) then we obtain

$$k(x, x^{\sigma})x^{\Delta} = j(t).$$

Since (7.3) holds, we may replace k above and obtain

$$\int_0^1 K'(x(t) + h[x^{\sigma}(t) - x(t)]) \, dh \cdot x^{\Delta}(t) = j(t)$$

which simplifies to

$$\int_0^1 K'(x(t) + h\mu(t)x^{\Delta}(t)) \, dh \cdot x^{\Delta}(t) = j(t).$$

By the chain rule we have

$$[K(x(t))]^{\Delta} = j(t).$$

Thus, taking the delta integral of both sides in the previous expression and incorporating the initial condition we obtain

$$K(x(t)) = \int_a^t j(s) \, \Delta s + K(A).$$

□

We now present a simple example.

Example 7.3 *Consider the dynamic IVP*

$$\begin{aligned} x^{\Delta} &= \frac{e_{\alpha}(t,0)}{x+x^{\sigma}}, \quad t \in [0,b]_{\mathbb{T}}; \\ x(0) &= A > 0; \end{aligned}$$

where α is a positive constant. We claim that this dynamic IVP has a solution for arbitrary $\mathbb{T}$.

Proof We will use Theorem 7.2. See that our dynamic equation is separable with

$$k(x, x^{\sigma}) = x + x^{\sigma} \quad \text{and} \quad j(t) = e_{\alpha}(t,0).$$

If we choose K to be $K(v) := v^2$ then we see $K'(v) = 2v$ and

$$\begin{aligned} \int_0^1 K'(x + h[x^{\sigma} - x]) \; dh &= \int_0^1 2(x + h[x^{\sigma} - x]) \; dh \\ &= x + x^{\sigma} = k(x, x^{\sigma}). \end{aligned}$$

Hence

$$[x(t)]^2 = \int_a^t e_{\alpha}(s,0) \; \Delta s + A^2 = \frac{e_{\alpha}(t,0) - 1}{\alpha} + A^2$$

and so

$$x(t) = \left[\frac{e_{\alpha}(t,0) - 1}{\alpha} + A^2 \right]^{1/2}.$$

The result now follows from Theorem 4.8. □

8 Open problems and remarks

We conclude this work with some suggestions for possible further investigations resulting from the ideas of this paper.

For a positive constant β, the new TZ–norm $\| \cdot \|_{\beta}$ and new TZ–metric d_{β} introduced in Section 4 seem to have potential applications to a wide range of additional areas. In particular, the use of the above norm and metric over the traditional sup–norm and sup–metric may lead to further advancements in local existence, uniqueness and approximation of solutions to: Volterra integral equations of the second kind on time scales; integro–differential equations of Volterra type on time scales; delay–dynamic equations on time scales; and so on. In fact, there appear to be a host of areas that use or rely on Banach's fixed–point theorem to gain important properties of solutions to various types of equations and, as such, existing ideas might be simplified by adopting the TZ–norm and TZ–metric. This may include, for example: continuous dependence of solutions to dynamic equations on time scales with respect to initial values and parameters; and nonlinear variation of parameters.

As a result of the convergence conclusions of Theorems 4.3 and 4.5, we feel that it would be interesting to go further and to systematically and explicitly compute or approximate solutions to dynamic equations on time scales.

Many of the ideas from this work could possibly be extended to the area of dynamic equations on "measure chains", which is a generalisation of the field of time scales, see [5, Chap.8], by using the ideas in Sections 3 and 5.

In view of the ideas raised in Sections 6 and 7 regarding the generalised dynamic equation

$$x^{\Delta} = f(t, x, x^{\sigma}), \tag{8.1}$$

we believe that it might be interesting to formulate new qualitative and quantitative results for (8.1).

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