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On first-order discrete boundary value problems

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This article analyzes a nonlinear system of first-order difference equations with periodic and non-periodic boundary conditions. Some sufficient conditions are presented under which: potential solutions to the equations will satisfy certain *a priori* bounds; and the equations will admit at least one solution. The methods involve new dynamic inequalities and use of Brouwer degree theory. The new results are compared with those featuring in the theory of solutions to boundary value problems for differential equations.

Keywords: Existence of solutions; Boundary value problems; Difference equations; Brouwer degree theory

AMS Subject Classification: 39A12

1. Introduction

The continuing interest in the field of difference equations can be attributed to two main factors:

1. due to the theory's powerful and versatile applications to almost all areas of science, engineering and technology, which teem with discrete phenomena; and
2. from the emergence and popularity of computers, where differential equations are solved by utilizing their approximative difference-equation formulations.

Therefore, scientific advancements in the area of difference equations are naturally motivated, and are of significant interest. The valuable uses of difference equations lead to a deeper theoretical analysis of the subject. The theory of the *qualities of solutions to difference equations* are particularly interesting, as studies have shown rich distinctions and interesting links between the qualitative theory of solutions to difference equations and the qualitative theory of solutions to differential equations. This has included essential concepts such as: existence, uniqueness and multiplicity of solutions; existence and non-existence of spurious solutions; *a priori* bounds on solutions; stability, instability and disconjugacy of solutions. We refer the reader to [1–13,15–27], for some previous insights in these directions.

This paper considers *a priori* bounds on, and the existence of, solutions to the following first-order difference equations:

$$\Delta x(t) = f(t, x(t)), \quad t \in [0, N]_{\mathbb{Z}} := [0, N] \cap \mathbb{Z}, \tag{1}$$

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$$Mx(0) + Rx(N+1) = 0, \quad 0 < N \in \mathbb{Z} \quad M + R \neq 0; \text{ and} \tag{2}$$

$$\Delta x(t) + b(t)x(t) = g(t, x(t)), \quad t \in [0, N]_{\mathbb{Z}}, \tag{3}$$

$$x(0) = x(N+1), \tag{4}$$

Above: f and $g : [0, N]_{\mathbb{Z}} \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous, nonlinear functions; $\Delta x(t) := x(t+1) - x(t)$ is the usual forward difference; $b : [0, N]_{\mathbb{Z}} \to \mathbb{R}$ is continuous; M and R are given constants in $\mathbb{R}$; and N is a given positive constant in $\mathbb{Z}$. Equation (1) subject to equation (2) is known as a discrete boundary value problem (BVP) with non-periodic boundary conditions. Equation (3) subject to equation (4) is known as a discrete BVP with periodic boundary conditions.

The article is organised as follows. Section 2 deals with *a priori* bounds on solutions to the above equations. The discrete BVPs are reformulated as equivalent summation equations. Then, novel conditions are introduced, involving the right-hand side of the difference equations and also the boundary conditions, so that all potential solutions to the BVPs are bounded *a priori*. Section 3 is devoted to the existence of solutions. The *a priori* bounds from Section 2 are applied, in conjunction with fixed-point operator theory and Brouwer degree, so that at least one solution to the discrete BVP of interest will exist.

Examples to highlight the theory are presented throughout the paper and a comparison is made between the new results and the qualitative theory of solutions to differential equations. The new results contained herein are given for systems of equations, however, the results are also new even for scalar difference equations.

A solution to equations (1) and (2) is a $x : [0, N+1]_{\mathbb{Z}} \to \mathbb{R}^n$ (which may also be equivalently denoted by $\{x(t)\}_{t=0}^{N+1} \in \mathbb{R}^{(N+2)n}$) that satisfies equations (1) and (2). A solution to equations (3) and (4) is defined similarly.

A tool used in this paper to gain existence of solutions is Brouwer degree theory and the interested reader is referred to [14] for more information.

2. *A priori* bounds on solutions

In what follows, if $y, z \in \mathbb{R}^n$ then $\langle y, z \rangle$ denotes the usual inner product and $\|z\|$ denotes Euclidean norm of z on $\mathbb{R}^n$.

We will need to use the following identities involving products and sums involving arbitrary functions $a : \mathbb{Z} \to \mathbb{R}$

$$\prod_{s=0}^{-1} a(s) = 1, \quad \sum_{s=0}^{-1} a(s) = 0.$$

LEMMA 2.1. *Suppose*

$$M + R \neq 0. \tag{5}$$

The discrete BVP equations (1) *and* (2) *is equivalent to the summation equation*

$$x(t) = \sum_{s=0}^{t-1} f(s, x(s)) - (M+R)^{-1} R \sum_{s=0}^{N} f(s, x(s)), \quad t \in [0, N+1]_{\mathbb{Z}}. \tag{6}$$

Proof. Let $x : [0, N]_{\mathbb{Z}} \to \mathbb{R}^n$ satisfy equations (1) and (2). It is easy to see that

$$x(t) = x(0) + \sum_{s=0}^{t-1} f(s, x(s)), \quad t \in [0, N+1]_{\mathbb{Z}}, \tag{7}$$

and

$$x(N+1) = x(0) + \sum_{s=0}^{N} f(s, x(s)). \tag{8}$$

So equation (2) gives

$$0 = Mx(0) + R\left(x(0) + \sum_{s=0}^{N} f(s, x(s))\right) \tag{9}$$

and rearranging equation (9) we obtain

$$x(0) = -(M+R)^{-1} R \sum_{s=0}^{N} f(s, x(s)). \tag{10}$$

So substituting equation (10) into equation (7) we obtain equation (6). If x is a solution to equation (6) then is it easy to show that equations (1) and (2) hold by direct calculation. □

LEMMA 2.2. *Suppose*

$$1 - b \text{ has no zeros in } [0, N]_{\mathbb{Z}}, \quad \text{and } \prod_{s=0}^{N} [1 - b(s)] \neq 1. \tag{11}$$

The discrete BVP equations (3) *and* (4) *is equivalent to the summation equation*

$$\begin{aligned} x(t) = {} & \prod_{s=0}^{t-1} [1 - b(s)] \\ & \times \left[\frac{\prod_{s=0}^{N} [1 - b(s)]}{1 - \prod_{s=0}^{N} [1 - b(s)]} \sum_{i=0}^{N} \frac{g(i, x(i))}{\prod_{s=0}^{i} [1 - b(s)]} + \sum_{i=0}^{t-1} \frac{g(i, x(i))}{\prod_{s=0}^{i} [1 - b(s)]} \right] \end{aligned} \tag{12}$$

for each $t \in [0, N+1]_{\mathbb{Z}}$.

Proof. Let x be a solution to equations (3) and (4). By the discrete quotient rule [13] we have

$$\begin{aligned} \Delta\left[\frac{x(t)}{\prod_{s=0}^{t-1} [1 - b(s)]}\right] &= \frac{[\Delta x(t)] \prod_{s=0}^{t-1} [1 - b(s)] - x(t)(-b(t)) \prod_{s=0}^{t-1} [1 - b(s)]}{\prod_{s=0}^{t} [1 - b(s)] \prod_{s=0}^{t-1} [1 - b(s)]} \\ &= \frac{\Delta x(t) + x(t) b(t)}{\prod_{s=0}^{t} [1 - b(s)]} = \frac{g(t, x(t))}{\prod_{s=0}^{t} [1 - b(s)]}. \end{aligned}$$

Summing both sides above, obtain

$$\sum_{i=0}^{t-1} \Delta\left[\frac{x(i)}{\prod_{s=0}^{i-1} [1 - b(s)]}\right] = \sum_{i=0}^{t-1} \frac{g(i, x(i))}{\prod_{s=0}^{i} [1 - b(s)]}$$

and rearrange to obtain

$$x(t) = \prod_{s=0}^{t-1}[1-b(s)]\left[x(0) + \sum_{i=0}^{t-1}\frac{g(i,x(i))}{\prod_{s=0}^{i}[1-b(s)]}\right] \tag{13}$$

for each $t \in [0, N+1]_{\mathbb{Z}}$.

Now using the boundary conditions and equation (13) obtain

$$x(0) = x(N+1) = \prod_{s=0}^{N}[1-b(s)]\left[x(0) + \sum_{i=0}^{N}\frac{g(i,x(i))}{\prod_{s=0}^{i}[1-b(s)]}\right]$$

and a rearrangement gives

$$x(0) = \frac{\prod_{s=0}^{N}[1-b(s)]}{1-\prod_{s=0}^{N}[1-b(s)]}\sum_{i=0}^{N}\frac{g(i,x(i))}{\prod_{s=0}^{i}[1-b(s)]}$$

which is substituted into equation (13) and the result follows.

If x is a solution to equation (12) then direct computation shows that equations (3) and (4) both hold. □

Some new *a priori* bound results are now presented.

THEOREM 2.3. *Suppose equation* (5) *holds and* $f \in C([0,N]_{\mathbb{Z}} \times \mathbb{R}^n; \mathbb{R}^n)$. *If there exist non-negative constants* α *and* K *such that*

$$\|f(t,q)\| \leq \alpha[2\langle q, f(t,q)\rangle + \|f(t,q)\|^2] + K, \ \textit{for all} \ (t,q) \in [0,N]_{\mathbb{Z}} \times \mathbb{R}^n, \tag{14}$$

$$\textit{and} \ \left|\frac{M}{R}\right| \leq 1, \tag{15}$$

then all possible solutions to the BVP equations (1) *and* (2) *satisfy* $\|x(t)\| \leq Q$ *for* $t \in [0, N+1]_{\mathbb{Z}}$, *where* Q *is a non-negative constant involving M, R, N and K.*

Proof. Let x be a solution to equations (1) and (2) and define a constant P by

$$P = (1 + |(M+R)^{-1}R|). \tag{16}$$

From Lemma 2.1, x must also satisfy equation (6). Consider $\|x(t)\|^2$ for all $t \in [0, N+1]_{\mathbb{Z}}$. By the discrete product rule [13, Theorem 2.2 (d)] we have

$$\begin{aligned}\Delta_t(\|x(t)\|^2) &= \langle x(t) + x(t+1), \Delta x(t)\rangle, \ \ t \in [0,N]_{\mathbb{Z}},\\ &= 2\langle x(t), \Delta x(t)\rangle + \|\Delta x(t)\|^2, \ \ \text{since} \ \ x(t+1)\\ &= \Delta x(t) + x(t) = 2\langle x(t), f(t,x(t))\rangle + \|f(t,x(t))\|^2.\end{aligned} \tag{17}$$

Notice also that equations (15) and (2) together imply that $\|x(N+1)\| \leq \|x(0)\|$.

So from equation (6) we have for $t \in [0, N+1]_{\mathbb{Z}}$,

$$\begin{aligned}\|x(t)\| &= \left\| \sum_{s=0}^{t-1} f(s, x(s)) - (M+R)^{-1} R \sum_{s=0}^{N} f(s, x(s)) \right\| \leq P \sum_{i=0}^{N} \|f(i, x(i))\| \\ &\leq P \sum_{i=0}^{N} (\alpha[2\langle x(i), f(i, x(i))\rangle + \|f(i, x(i))\|^2] + K) \\ &= P \sum_{i=0}^{N} [\alpha \Delta_i(\|x(i)\|^2) + K], \ \text{ from } \ (17) \\ &= P[\alpha(\|x(N+1)\|^2 - \|x(0)\|^2) + K(N+1)] \leq PK(N+1).\end{aligned}$$

Hence the bound on all possible solutions follows. □

Note that equation (5) means that Theorem 2.3 does not apply to the BVPs equations (3) and (4) or equations (1) and (4). The next two theorems will cover these outstanding cases.

THEOREM 2.4. *Suppose equation* (11) *holds and* $g \in C([0, N]_{\mathbb{Z}} \times \mathbb{R}^n; \mathbb{R}^n)$. *If there exist non-negative constants* α *and* K *such that*

$$\begin{aligned}\frac{\|g(t,p)\|}{\left|\prod_{s=0}^{t} [1 - b(s)]\right|} &\leq \alpha\left[2\left(\langle p, g(t,p)\rangle - b(t)\|p\|^2\right) + \|g(t,p) - b(t)p\|^2\right] + K, \\ &\qquad \textit{for all } \ (t,p) \in [0,N]_{\mathbb{Z}} \times \mathbb{R}^n,\end{aligned} \tag{18}$$

then all possible solutions to the BVP equations (3) *and* (4) *satisfy* $\|x(t)\| \leq Q_1$ *for* $t \in [0, N+1]_{\mathbb{Z}}$, *where* Q_1 *is a non-negative constant involving b, N and K.*

Proof. From Lemma 2.2, consider the equation equivalent to equations (3) and (4) given by equation (12). Define the positive constant P_1 by

$$P_1 = \max_{t \in [0, N+1]_{\mathbb{Z}}} \left| \prod_{s=0}^{t-1} [1 - b(s)] \left[1 + \frac{\left|\prod_{s=0}^{N} [1 - b(s)]\right|}{\left|1 - \prod_{s=0}^{N} [1 - b(s)]\right|} \right] \right|. \tag{19}$$

Let x be a solution to equations (3) and (4) and consider $\|x(t)\|^2$ for all $t \in [0, N+1]_{\mathbb{Z}}$. By the discrete product rule we have

$$\begin{aligned}\Delta_t(\|x(t)\|^2) &= 2\langle x(t), \Delta x(t)\rangle + \|\Delta x(t)\|^2, \ \ t \in [0, N]_{\mathbb{Z}}, \\ &= 2\langle x(t), g(t, x(t) - b(t)x(t))\rangle + \|g(t, x(t)) - b(t)x(t)\|^2.\end{aligned} \tag{20}$$

If we take norms in equation (12) then for $t \in [0, N+1]_{\mathbb{Z}}$,

$$\begin{aligned}
\|x(t)\| &\leq P_1 \sum_{i=0}^{N} \frac{\|g(i, x(i))\|}{\left|\prod_{s=0}^{i}[1-b(s)]\right|} \\
&\leq P_1 \sum_{i=0}^{N} \left(\alpha\left[2\langle x(i), g(i, x(i)) - b(i)x(i)\rangle + \|g(i, x(i)) - b(i)x(i)\|^2\right] + K\right) \\
&= P_1 \sum_{i=0}^{N} [\alpha \Delta_i(\|x(i)\|^2) + K], \quad \text{from (20)} \\
&= P_1[\alpha(\|x(N+1)\|^2 - \|x(0)\|^2) + K(N+1)] = P_1 K(N+1).
\end{aligned}$$

Hence the bound on all possible solutions follows. □

Attention now turns to the BVP equations (1) and (4).

THEOREM 2.5. *Let* $f \in C([0, N]_{\mathbb{Z}} \times \mathbb{R}^n; \mathbb{R}^n)$. *If there exist non-negative constants* α *and* K *such that*

$$\frac{\|f(t,p) - p\|}{2^{t+1}} \leq \alpha\left[2(\langle p, f(t,p)\rangle + \|f(t,p)\|^2\right] + K, \ \ \textit{for all} \ \ (t,p) \in [0, N]_{\mathbb{Z}} \times \mathbb{R}^n,$$

then all possible solutions to the BVP equations (1) *and* (4) *satisfy* $\|x(t)\| \leq Q_2$ *for* $t \in [0, N+1]_{\mathbb{Z}}$, *where* Q_2 *is a non-negative constant involving N and K.*

Proof. Rewrite the BVP equations (1) and (4) in the form

$$\Delta x(t) - x(t) = f(t, x(t)) - x(t), \quad t \in [0, N]_{\mathbb{Z}}, \quad x(0) = x(N+1),$$

so that it becomes of the type equations (3) and (4) with $b = -1$ and $g(t,p) = f(t,p) - p$. It is not difficult to show that $\prod_{s=0}^{t}[1 - b(s)] = 2^{t+1}$ and thus equation (11) holds. If (2.5) holds then equation (18) holds and the result follows from Theorem 2.4. □

Some examples are now presented to highlight the theory.

Example 2.6. Consider the discrete, scalar BVP ($n = 1$) given by

$$\Delta x(t) = [x(t)]^3 + x(t) + t, \quad t \in [0, 99]_{\mathbb{Z}}, \tag{21}$$

$$x(0) = x(100). \tag{22}$$

Let $f(t,p) = p^3 + p + t$ and see that $|f(t,p) - p| \leq |p^3| + 100$ for $(t,p) \in [0, 99]_{\mathbb{Z}} \times \mathbb{R}$. For α and K to be chosen below, see that

$$\begin{aligned}
\alpha[2pf(t,p) + (f(t,p))^2] + K &= \alpha[2(p^4 + p^2 + pt) + (p^3 + p + t)^2] + K \\
&\geq 2\alpha(p^4 + p^2 + pt) + K = (p^4 + p^2 + pt) + 400, \\
&\text{for the choices } \alpha = \frac{1}{2},\ K = 400 \geq |p^3| + 100 \\
&\geq |f(t,p) - p| \geq \frac{|f(t,p) - p|}{2^{t+1}}, \ \text{for all } (t,p) \in [0, 99]_{\mathbb{Z}} \times \mathbb{R}
\end{aligned}$$

and thus (2.5) holds. Thus, all of the conditions of Theorem 2.5 hold and all possible solutions to the discrete BVP equations (21) and (22) satisfy an *a priori* bound. □

Example 2.7. Consider the discrete, scalar BVP ($n = 1$) given by

$$\Delta x(t) = [x(t) - 1]^2 + t, \quad t \in [0, N]_{\mathbb{Z}}, \tag{23}$$

$$x(0) + x(N+1) = 0. \tag{24}$$

Let $f(t,q) = (q-1)^2 + t$ and see that $|f(t,q)| \leq (q-1)^2 + N$. For α and K to be chosen below, see that

$$\begin{aligned} \alpha[2qf(t,q) + (f(t,q))^2] + K &= \alpha[2q + f(t,q)]f(t,q) + K \\ &= \alpha(q^2 + 1 + t)[(q-1)^2 + t] + K \geq \alpha(q-1)^2 + K \\ &= (q-1)^2 + N, \text{ for the choice } \alpha = 1, \ K = N \\ &\geq |f(t,q)|, \text{ for all } (t,q) \in [0,N]_{\mathbb{Z}} \times \mathbb{R}^n \end{aligned}$$

and thus equation (14) holds for the choices $\alpha = 1$ and $K = N$. Since $M = 1 = R$, it is easy to see that equation (15) holds. Thus, all of the conditions of Theorem 2.3 hold and all solutions to the discrete BVP equations (23) and (24) must satisfy an *a priori* bound. □

Remark 2.8. It is easy to see that if there exist non-negative constants α and K such that

$$\|f(t,q)\| \leq 2\alpha\langle q, f(t,q)\rangle + K, \text{ for all } (t,q) \in [0,N]_{\mathbb{Z}} \times \mathbb{R}^n, \tag{25}$$

then equation (14) holds. Thus, if equation (14) is replaced by equation (25) then under these stronger conditions, Theorem 2.3 ensures that all solutions to the discrete BVP equations (1) and (2) will satisfy an *a priori* bound.

Condition (25) has been used in guaranteeing *a priori* bounds on solutions to the BVP [28,29]

$$x' = f(t,x), \quad t \in [a,c], \tag{26}$$

$$Mx(0) + Rx(N) = 0. \tag{27}$$

For discrete BVPs, however, it is of more benefit to consider the less restrictive inequality (14) rather than equation (25). For example, the right-hand side of equation (23) satisfies equation (14) for the choices $\alpha = 1$ and $K = N$, but the right-hand side of equation (23) cannot satisfy equation (25) for any choice of non-negative α and K. Hence, Example 2.7 illustrates a distinction between the qualitative theories of solutions to difference and differential equations.

3. Existence

In this section the *a priori* bounds from Section 2 are applied to obtain some new existence results.

THEOREM 3.1. *Suppose equation* (5) *holds and* $f \in C([0,N]_{\mathbb{Z}} \times \mathbb{R}^n; \mathbb{R}^n)$. *If there exist non-negative constants* α *and* K *such that*

$$\|f(t,p)\| \leq 2\alpha\langle p, f(t,p)\rangle + K, \text{ for all } (t,p) \in [0,N]_{\mathbb{Z}} \times \mathbb{R}^n, \tag{28}$$

and equation (15) *holds then the BVP equations* (1) *and* (2) *have at least one solution.*

Proof. Consider the map $T : \mathbb{R}^{(N+2)n} \to \mathbb{R}^{(N+2)n}$ defined by

$$T(x(t)) := -(M+R)^{-1} R \sum_{s=0}^{N} f(s, x(s)) + \sum_{s=0}^{t-1} f(s, x(s)), \tag{29}$$

for all $t \in [0, N+1]_{\mathbb{Z}}$. Thus, our problem is reduced to proving the existence of at least one v such that

$$v = Tv. \tag{30}$$

Since f is continuous, see that T is also a continuous map. In order to invoke the non-zero property of Brouwer degree under homotopy invariance, it will be shown that all solutions to

$$x = \lambda Tx, \quad \lambda \in [0,1], \tag{31}$$

satisfy

$$x \neq \lambda Tx, \quad \forall x \in \partial\Omega, \quad \forall \lambda \in [0,1], \tag{32}$$

where Ω is a suitable ball containing the origin. Let

$$\Omega := \left\{ x : [0, N+1]_{\mathbb{Z}} \to \mathbb{R}^n \,\middle|\, \max_{t \in [0,N+1]_{\mathbb{Z}}} \|x(t)\| < PK(N+1) + 1 \right\} \subset \mathbb{R}^{(N+2)n}$$

where P is defined in equation (16). Notice that equation (31) is equivalent to the BVP

$$\Delta x(t) = \lambda f(t, x(t)), \quad t \in [0,N]_{\mathbb{Z}}, \; \lambda \in [0,1], \tag{33}$$

$$Mx(0) + Rx(N+1) = 0, \quad 0 < N \in \mathbb{Z}, \; M + R \neq 0. \tag{34}$$

If equation (28) holds then for each $(t, p, \lambda) \in [0,N]_{\mathbb{Z}} \times \mathbb{R}^n \times [0,1]$ we have

$$\|\lambda f(t,p)\| \leq 2\alpha\langle p, \lambda f(t,p)\rangle + \lambda K \leq \alpha[2\langle p, \lambda f(t,p)\rangle + \|\lambda f(t,p)\|^2] + K$$

so that Lemma 2.1 applies to equations (33) and (34). Hence all solutions x to equations (33) and (34) must satisfy $x \in \Omega$ and thus equation (32) holds.

By the non-zero property of Brouwer degree under homotopy, equation (30) must have at least one solution in Ω and hence the BVP equations (1) and (2) must have at least one solution. □

THEOREM 3.2. *Let equation* (11) *hold,* $g \in C([0,N]_{\mathbb{Z}} \times \mathbb{R}^n; \mathbb{R}^n)$ *and* $b \in C([0,N]_{\mathbb{Z}}; (-\infty, 0))$. *If there exist non-negative constants* α *and* K *such that*

$$\frac{\|g(t,p)\|}{\left|\prod_{s=0}^{t} [1 - b(s)]\right|} \leq 2\alpha\langle p, g(t,p)\rangle + K, \quad \text{for all } (t,p) \in [0,N]_{\mathbb{Z}} \times \mathbb{R}^n, \tag{35}$$

then the BVP equations (3) *and* (4) *has at least one solution.*

Proof. Consider the map $U : \mathbb{R}^{(N+2)n} \to \mathbb{R}^{(N+2)n}$ defined by

$$Ux(t) = \prod_{s=0}^{t-1}[1-b(s)]\left[\frac{\prod_{s=0}^{N}[1-b(s)]}{1-\prod_{s=0}^{N}[1-b(s)]}\sum_{i=0}^{N}\frac{g(i,x(i))}{\prod_{s=0}^{i}[1-b(s)]} + \sum_{i=0}^{t-1}\frac{g(i,x(i))}{\prod_{s=0}^{i}[1-b(s)]}\right] \tag{36}$$

for all $t \in [0, N+1]_{\mathbb{Z}}$. Thus, our problem is reduced to proving the existence of at least one q such that

$$q = Uq. \tag{37}$$

Since g and b are continuous, see that U is also a continuous map. Consider the family of problems

$$x = \lambda Ux, \quad \lambda \in [0,1]. \tag{38}$$

In order to invoke the non-zero property of Brouwer degree under homotopy invariance, it will be shown that all solutions to equation (38) satisfy

$$x \neq \lambda Ux, \quad \forall x \in \partial\Omega_1, \quad \forall \lambda \in [0,1], \tag{39}$$

where Ω_1 is a suitable ball containing the origin. Let

$$\Omega_1 := \left\{ x : [0,N]_{\mathbb{Z}} \to \mathbb{R}^n \middle| \max_{t\in[0,N+1]_{\mathbb{Z}}} \|x(t)\| < P_1K(N+1)+1 \right\} \subset \mathbb{R}^{(N+2)n}$$

where P_1 is defined in the proof of Theorem 2.4. Note that equation (38) is equivalent to the BVP

$$\Delta x(t) + b(t)x(t) = \lambda g(t,x(t)), \quad t \in [0,N]_{\mathbb{Z}}, \ \lambda \in [0,1], \tag{40}$$

$$x(0) = x(N+1). \tag{41}$$

Using a similar argument to the proof of Theorem 3.1 it is not difficult to show that if equation (35) holds then Lemma 2.4 applies to equations (40) and (41). Hence all solutions x to equations (40) and (41) satisfy $x \in \Omega_1$ and thus equation (39) holds. The non-zero Brouwer degree under homotopy invariance ensures the existence of at least one solution in Ω_1 to equation (37) and hence equations (3) and (4) have at least one solution. □

Attention now turns to the BVP equations (1) and (4).

THEOREM 3.3. *Let* $f \in C([0,N]_{\mathbb{Z}} \times \mathbb{R}^n; \mathbb{R}^n)$. *If there exist non-negative constants* α *and* K *such that*

$$\frac{\|f(t,p)-p\|}{2^{t+1}} \le 2\alpha\langle p, f(t,p)\rangle + K, \ \textit{for all} \ (t,p) \in [0,N]_{\mathbb{Z}} \times \mathbb{R}^n \times [0,1], \tag{42}$$

then the BVP equations (1) *and* (4) *have at least one solution.*

Proof. Rewrite the BVP equations (1) and (4) in the form

$$\Delta x(t) - x(t) = f(t,x(t)) - x(t), \quad t \in [0,N]_{\mathbb{Z}}, \quad x(0) = x(N+1),$$

so that it becomes of the type equations (3) and (4) with $b = -1$ and $g(t,p) = f(t,p) - p$. For the above special cases, if equation (42) then equation (35) holds and the result follows from Theorem 3.2. $\square$

Some examples are now presented to highlight the theory.

Example 3.4 Consider the discrete, scalar BVP ($n = 1$) given by equations (21) and (22). Let $f(t,p) = p^3 + p + t$. Under similar working to that in Example 2.6 we can choose $\alpha = 1/2$ and $K = 400$ so that equation (42) holds. Thus, by Theorem 3.3 the discrete BVP equations (21) and (22) have at least one solution. $\square$

Example 3.5 Consider equations (1) and (2) with $n = 2$ and f given by

$$\begin{aligned} f(t,p) &= (h(t,y,z), j(t,y,z)), \quad t \in [0,N]_{\mathbb{Z}}, \\ &= \left((t+1)y^3 + y\mathrm{e}^{-z^2} + 1, (t+1)z^3 + z\mathrm{e}^{-y^2}\right). \end{aligned} \tag{43}$$

The above f satisfies the conditions of Theorem 3.1. Note that for all $(t,p) \in [0,N]_{\mathbb{Z}} \times \mathbb{R}^2$ we have

$$\|f(t,p)\| \leq |h(t,y,z)| + |j(t,y,z)| \leq (N+1)|y|^3 + |y|\mathrm{e}^{-z^2} + (N+1)|z|^3 + |z|\mathrm{e}^{-y^2} + 1.$$

Below, we will need the following simple inequalities:

$$\begin{gathered} w^4 \geq |w|^3 - 1, w^4 + w \geq |w|^3 - 10, \ \text{for all } \ w \in \mathbb{R}, \\ b^2\mathrm{e}^{-a^2} \geq |b|\mathrm{e}^{-a^2} - 1, \ \text{for all } \ (a,b) \in \mathbb{R}^2. \end{gathered}$$

For $\alpha \geq 0$ and $K \geq 0$ to be chosen below, consider for $(t,p) \in [0,N]_{\mathbb{Z}} \times \mathbb{R}^2$,

$$\begin{aligned} 2\alpha\langle p, f(t,p)\rangle + K &\geq 2\alpha\left[y^4 + y + y^2\mathrm{e}^{-z^2} + z^4 + z^2\mathrm{e}^{-y^2}\right] + K \\ &\geq 2\alpha\left[|y|^3 - 10 + |y|\mathrm{e}^{-z^2} - 1 + |z|^3 - 1 + |z|\mathrm{e}^{-y^2} - 1\right] + K \\ &\geq (N+1)|y|^3 + |y|\mathrm{e}^{-z^2} + (N+1)|z|^3 + |z|\mathrm{e}^{-y^2} + 1 \\ &\geq \|f(t,p)\| \ \text{choosing} \ \alpha = (N+1)/2, \quad K = 13N + 14. \end{aligned}$$

Thus f satisfies the conditions of Theorem 3.1 for the choices $\alpha = (N+1)/2$ and $K = 13N + 14$.

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References

[1] Agarwal, R.P., Cabada, A., Otero-Espinar and Dontha, S., 2004, Existence and uniqueness of solutions for antiperiodic difference equations. *Archives of Inequalities and Applications*, **2**, 397–411.

[2] Agarwal, R.P., O'Regan, D. and Wong, P.J.Y., 2005, Existence of sign-constant solutions to a system of difference equations; the semipositone and singular case. *Journal of Difference Equations and Applications*, **11**, 151–171.

[3] Agarwal, R.P., Thompson, H.B. and Tisdell, C.C., 2003, Difference equations in Banach spaces. Advances in difference equations, IV. *Computers and Mathematics with Applications*, **45**, 1437–1444.

[4] Agarwal, R.P., Thompson, H.B. and Tisdell, C.C., 2003, Three-point boundary value problems for second-order discrete equations. Advances in difference equations, IV. *Computers and Mathematics with Applications*, **45**, 1429–1435.

[5] Amster, P., Rogers, C. and Tisdell, C.C., 2005, Existence of solutions to boundary value problems for dynamic systems on time scales. *Journal Mathematical Analysis and Applications*, **308**(2), 565–577.

[6] Anderson, D., Avery, R. and Henderson, J., 2004, Existence of solutions for the one dimensional p-Laplacian on time-scales. *Journal of Difference Equations and Applications*, **10**, 889–896.

[7] Atici, F.M., Cabada, A. and Ferreiro, J.B., 2002, Existence and comparison results for first order periodic implicit difference equations with maxima. *Journal of Difference Equations and Applications*, **8**, 357–369.

[8] Cheung, W.-S., 2004, Some discrete nonlinear inequalities and applications to boundary value problems for difference equations. *Journal of Difference Equations and Applications*, **10**, 213–233.

[9] Drábek, P., Thompson, H.B. and Tisdell, C., 2001, Multipoint boundary value problems for discrete equations. *Comment on Mathematical University of Carolin*, **42**(3), 459–468.

[10] Erbe, L., Peterson, A. and Tisdell, C.C., 2006, Monotone solutions to dynamic systems on time scales. *Journal of Difference Equations and Applications*, **12**(3–4), 343–355.

[11] Erbe, L., Peterson, A. and Tisdell, C.C., 2005, Existence of solutions to second-order BVPs on time scales. *Applicable Analysis*, **84**(10), 1069–1078.

[12] Kaufmann, E.R. and Kosmatov, N., 2004, Singular conjugate boundary value problems on a time scale. *Journal of Difference Equations and Applications*, **10**, 119–127.

[13] Kelley, W.G. and Peterson, A.C., 2001, *Difference Equations. An Introduction with Applications* (San Diego: Academic Press).

[14] Lloyd, N.G., 1978, *Degree Theory* (New York, NY: Cambridge University Press).

[15] Ma, R. and Raffoul, Y.N., 2004, Positive solutions of three-point nonlinear discrete second order boundary value problem. *Journal of Difference Equations and Applications*, **10**, 129–138.

[16] Mawhin, J., Thompson, H.B. and Tonkes, E., 2004, Uniqueness for boundary value problems for second order difference equations. *Journal of Difference Equations and Applications*, **10**, 749–757.

[17] Mawhin, J. and Tisdell, C.C., 2003, A note on the uniqueness of solutions to nonlinear, discrete, vector boundary value problems. *Nonlinear Analysis and Applications: to V. Lakshmikantham on his 80th Birthday* (Dordrecht: Kluwer Academic Publisher), **1, 2**, pp. 789–798.

[18] Murty, K.N. and Rao, C.V., 1999, Two-point boundary value problems associated with first order non-linear difference system—existence and uniqueness. *Applied Mathematics and Computation*, **100**, 177–188.

[19] Peterson, A.C., Raffoul, Y.N. and Tisdell, C.C., 2004, Three point boundary value problems on time scales. *Journal of Difference Equations and Applications*, **10**(9), 843–849.

[20] Peterson, A.C. and Tisdell, C.C., 2004, Boundedness and uniqueness of solutions to dynamic equations on time scales. *Journal of Difference Equations and Applications*, **10**, 1295–1306.

[21] Rachunkova, I. and Tisdell, C.C., 2006, Existence of non-spurious solutions to discrete boundary value problems. *Australian Journal of Mathematical Analysis and Applications*. **3**(2), 1–9.

[22] Rachunkova, I. and Tisdell, C.C., Existence of non-spurious solutions to discrete Dirichlet problems with lower and upper solutions. *Nonlinear Analysis*, doi: 10.1016/j.na.2006.07.010. In press, Available online 28 August 2006.

[23] Thompson, H.B. and Tisdell, C., 2000, Systems of difference equations associated with boundary value problems for second order systems of ordinary differential equations. *Journal of Mathematical Analysis and Applications*, **248**, 333–347.

[24] Thompson, H.B. and Tisdell, C., 2002, Boundary value problems for systems of difference equations associated with systems of second-order ordinary differential equations. *Applied Mathematics Letters*, **15**, 761–766.

[25] Thompson, H.B. and Tisdell, C.C., 2003, The nonexistence of spurious solutions to discrete, two-point boundary value problems. *Applied Mathematics Letters*, **16**(1), 79–84.

[26] Tian, Y. and Ge, W., 2006, Multiple positive solutions of boundary value problems for second-order discrete equations on the half-line. *Journal of Difference Equations and Applications*, **12**(2), 191–208.

[27] Tisdell, C.C., 2003, The uniqueness of solutions to discrete, vector, two-point boundary value problems. *Applied Mathematics Letters*, **16**, 1321–1328.

[28] Tisdell, C.C., 2005, Existence of solutions to first-order periodic boundary value problems. *Journal of Mathematical Analysis and Applications*, doi:10.1016/j.jmaa.2005.11.047. In press.

[29] Tisdell, C.C., 2006, On the solvability of nonlinear first-order boundary-value problems. *Electronic Journal of Differential Equations*, **2006**(80), 1–8.