

# ON BOUNDARY VALUE PROBLEMS FOR SECOND-ORDER DISCRETE INCLUSIONS

PETR STEHLÍK AND CHRISTOPHER C. TISDELL

Received 8 October 2004

We prove some existence theorems regarding solutions to boundary value problems for systems of second-order discrete inclusions. For a certain class of right-hand sides, we present some lemmas showing that all solutions to discrete second-order inclusions satisfy an *a priori* bound. Then we apply these *a priori* bounds, in conjunction with an appropriate fixed point theorem for inclusions, to obtain the existence of solutions. The theory is highlighted with several examples.

## 1. Introduction

The theory of differential inclusions has received much attention due to its versatility and generality. For example, differential inclusions can accurately model discontinuous processes, such as systems with dry friction; the work of an electric oscillator; and autopilot (and other) control systems [8]. When considering these (or other) situations in discrete time, the modeling process gives rise to a discrete (or difference) inclusion, rather than a differential inclusion. In many cases, considering the model in discrete time gives a more precise or realistic description [1].

Let  $X$  and  $Y$  be two normed spaces. A *set-valued map*  $G : X \rightarrow Y$  is a map that associates with any  $x \in X$  a set  $G(x) \subset Y$ . By  $CK(E)$ , we denote the set of nonempty, convex, and closed subsets of a Banach space  $E$ . We say that  $G : \mathbb{R}^n \rightarrow CK(\mathbb{R}^n)$  is *upper semicontinuous* if for all sequences  $\{u_i\} \subseteq \mathbb{R}^n$ ,  $\{v_i\} \subseteq \mathbb{R}^n$ , where  $i \in \mathbb{N}$ , the conditions  $u_i \rightarrow u_0$ ,  $v_i \rightarrow v_0$ , and  $v_i \in G(u_i)$  imply that  $v_0 \in G(u_0)$ . Since the upper semicontinuity plays an essential role in this paper, we illustrate this notion by the simple example [5, Example 4.1.1].

*Example 1.1.* The set-valued map  $f_1 : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$f_1(t) = \begin{cases} \{0\} & \text{for } t = 0, \\ [0, 1] & \text{for } t \in \mathbb{R} \setminus \{0\}, \end{cases} \quad (1.1)$$

is not upper semicontinuous. On the other hand, the set-valued map  $f_2 : \mathbb{R} \rightarrow \mathbb{R}$  defined

by

$$f_2(t) = \begin{cases} [0, 1] & \text{for } t = 0, \\ \{0\} & \text{for } t \in \mathbb{R} \setminus \{0\}, \end{cases} \quad (1.2)$$

is upper semicontinuous.

For more information about set-valued maps and differential inclusions, see Aubin and Cellina [3], Smirnov [8], or Erbe, Ma and Tisdell [6].

We are interested in the following boundary value problem (BVP) for second-order discrete inclusions:

$$\Delta^2 y(k-1) \in F(k, y(k), \Delta y(k)), \quad k = 1, \dots, T, \quad y(0) = A, \quad y(T+1) = B, \quad (1.3)$$

where  $A, B \in \mathbb{R}^d$  are constants and  $F : \{1, \dots, T\} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow CK(\mathbb{R}^d)$  is a set-valued map. A solution  $\bar{y} = \{y(k)\}_{k=0}^{T+1} \in \mathbb{R}^{(T+2)d}$  to (1.3) is a vector  $\bar{y} = \{y(0), \dots, y(T+1)\}$  such that each element  $y(k) \in \mathbb{R}^d$  satisfies the discrete inclusion for  $k = 1, \dots, T$  and the boundary conditions for  $k = 0$  and  $k = T + 1$ .

In Section 2, we show that under certain conditions on the right-hand side  $F$ , all solutions of (1.3) are bounded. The inequalities employed rely on growth conditions on  $F$  and on appropriate discrete maximum principles.

Section 3 contains the appropriate operator formulations for (1.3) to be considered as a fixed point problem.

In Section 4, we apply the results of Sections 2 and 3 to prove the existence of solutions to (1.3), in conjunction with the following fixed-point theorem [2, Theorem 1.2].

**THEOREM 1.2.** *Let  $E$  be a Banach space,  $U$  an open subset of  $E$ , and  $0 \in U$ . Suppose that  $P : \bar{U} \rightarrow CK(E)$  is an upper semicontinuous and compact map. Then either*

- (A1)  *$P$  has a fixed point in  $\bar{U}$ , or*
- (A2) *there exist  $u \in \partial U$  and  $\lambda \in (0, 1)$  with  $u \in \lambda P(u)$ .*

To prove the compactness of the image of an upper semicontinuous map, we will use a criterion which can be found in Berge [4, théorème VI.3].

**THEOREM 1.3.** *Let  $P : X \rightarrow Y$  be an upper semicontinuous map. If  $K$  is a compact set in  $X$ , then  $P(K)$  is a compact set in  $Y$ .*

In [2], Agarwal et al. gave conditions under which the following BVP has at least one solution:

$$\Delta^2 y(k-1) \in F(k, y(k)), \quad k = 1, \dots, T, \quad y(0) = 0, \quad y(T+1) = 0, \quad (1.4)$$

where  $F : \{1, \dots, T\} \times \mathbb{R} \rightarrow CK(\mathbb{R})$ . In comparison with the results and conditions in [2], we introduce new inequalities unrelated to those in [2] and we also extend some of the results in [2].

### 2. A priori bound

In this section, we prove two different *a priori* bound results for the following system of BVPs for second-order discrete inclusions:

$$\Delta^2 y(k-1) \in \lambda F(k, y(k), \Delta y(k)), \quad k = 1, \dots, T, \quad y(0) = \lambda A, \quad y(T+1) = \lambda B, \tag{2.1}$$

where  $\lambda \in [0, 1]$ .

The study of the above family of BVPs is motivated by the family of inclusions in Theorem 1.2,  $u \in \lambda P(u)$ .

We denote  $\langle \cdot, \cdot \rangle$  as the Euclidean inner product and by  $\| \cdot \|$  the Euclidean norm on  $\mathbb{R}^n$ .

LEMMA 2.1. *Let  $F : \{1, \dots, T\} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow CK(\mathbb{R}^d)$  be a set-valued map. If there exist constants  $\alpha \geq 0$  and  $K \geq 0$  such that*

$$\| \phi \| \leq \alpha(2\langle p, \phi \rangle + \|q\|^2) + K, \quad \text{for } k = 1, \dots, T \text{ and all } (p, q) \in \mathbb{R}^{2d}, \phi \in F(k, p, q), \tag{2.2}$$

then all solutions  $\bar{y}$  of BVP for the system of discrete inclusions (2.1) satisfy

$$\|y(k)\| < R, \quad k = 0, \dots, T+1, \tag{2.3}$$

for  $\lambda \in [0, 1]$ , and  $R$  is defined by

$$R := \alpha\beta^2 + \beta + K \frac{(T+1)^2}{8} + 1, \quad \beta := \max \{ \|A\|, \|B\| \}. \tag{2.4}$$

*Proof.* We suppose that  $\bar{y}$  is a solution of (2.1). Since we work on a discrete topology, every solution of (2.1) is a solution of a system of discrete BVPs,

$$\Delta^2 y(k-1) = \lambda \hat{f}(k, y(k), \Delta y(k)), \quad k = 1, \dots, T, \quad y(0) = \lambda A, \quad y(T+1) = \lambda B, \tag{2.5}$$

where  $\hat{f} : \{1, \dots, T\} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow CK(\mathbb{R}^d)$  is a single-valued function such that  $\hat{f}(k, p, q) \in F(k, p, q)$  for every  $k = 1, \dots, T$  and  $(p, q) \in \mathbb{R}^{2d}$ . In the theory of the set-valued map,  $\hat{f}$  is called a *selector* of  $F$ .

Then  $\bar{y}$  solves the summation equation

$$y(k) = \lambda \Phi(k) + \lambda \sum_{l=1}^T G(k, l) \hat{f}(l, y(l), \Delta y(l)), \quad k = 0, \dots, T+1, \tag{2.6}$$

where

$$\lambda \Phi(k) = \lambda \frac{A(T+1) + (B-A)k}{T+1}, \tag{2.7}$$

and  $G : \{0, \dots, T + 1\} \times \{1, \dots, T\} \rightarrow \mathbb{R}^d$  defined by

$$G(k, l) = \begin{cases} -\frac{1}{T+1}l(T+1-k), & l = 1, \dots, k-1, \\ -\frac{1}{T+1}k(T+1-l), & l = k, \dots, T, \end{cases} \quad (2.8)$$

is Green's function for the BVP,

$$\Delta^2 y(k-1) = 0, \quad k = 1, \dots, T, \quad y(0) = 0, \quad y(T+1) = 0. \quad (2.9)$$

Since  $\|\lambda\Phi(k)\| \leq \beta$  for each  $k = 0, \dots, T + 1$ , we obtain that

$$\|y(k)\| \leq \beta + \sum_{l=1}^T |G(k, l)| \lambda \|\hat{f}(l, y(l), \Delta y(l))\|. \quad (2.10)$$

Using (2.2), we have that

$$\begin{aligned} \|y(k)\| &\leq \beta + \sum_{l=1}^T |G(k, l)| \lambda \left\{ \alpha \left[ 2\langle y(l), \hat{f}(l, y(l), \Delta y(l)) \rangle + \|\Delta y(l)\|^2 \right] + K \right\} \\ &\leq \beta + \sum_{l=1}^T |G(k, l)| \left\{ \alpha \left[ 2\langle y(l), \lambda \hat{f}(l, y(l), \Delta y(l)) \rangle + \|\Delta y(l)\|^2 \right] + K \right\}. \end{aligned} \quad (2.11)$$

Define

$$r(k) := \|y(k)\|^2, \quad k = 0, \dots, T + 1, \quad (2.12)$$

and use the discrete product rule to calculate the second difference of  $r$  at the point  $k - 1$  to obtain

$$\Delta^2 r(k-1) = \|\Delta y(k)\|^2 + 2\langle y(k), \Delta^2 y(k-1) \rangle + \|\Delta y(k-1)\|^2. \quad (2.13)$$

By using the first equation in (2.5), we deduce that

$$\Delta^2 r(k-1) \geq \|\Delta y(k)\|^2 + 2\langle y(k), \lambda \hat{f}(k, y(k), \Delta y(k)) \rangle. \quad (2.14)$$

We install this into (2.11) to obtain that

$$\|y(k)\| \leq \beta + \alpha \sum_{l=1}^T |G(k, l)| \Delta^2 r(l-1) + \sum_{l=1}^T |G(k, l)| K. \quad (2.15)$$

Using (2.8) we make the following computations:

$$\begin{aligned}
 \sum_{l=1}^T |G(k,l)| \Delta^2 r(l-1) &= \frac{T+1-k}{T+1} \sum_{l=1}^{k-1} l \Delta^2 r(l-1) + \frac{k}{T+1} \sum_{l=k}^T (T+1-l) \Delta^2 r(l-1) \\
 &= \frac{T+1-k}{T+1} \left( [l \Delta r(l-1)]_1^k - \sum_{l=1}^{k-1} \Delta r(l) \right) \\
 &\quad + \frac{k}{T+1} \left( [(T+1-l) \Delta r(l-1)]_k^{T+1} + \sum_{l=k}^T \Delta r(l) \right) \tag{2.16} \\
 &= \frac{T+1-k}{T+1} (k \Delta r(k-1) - \Delta r(0) - r(k) + r(1)) \\
 &\quad + \frac{k}{T+1} (-(T+1-k) \Delta r(k-1) + r(T+1) - r(k)) \\
 &= \frac{T+1-k}{T+1} r(0) + \frac{k}{T+1} r(T+1) - r(k) \leq \beta^2.
 \end{aligned}$$

Finally, if we consider this estimation and (see, e.g., [7, Exercise 6.20])

$$\max_{k \in \{0, \dots, T+1\}} \sum_{l=1}^T |G(k,l)| \leq \frac{(T+1)^2}{8}, \tag{2.17}$$

we rewrite (2.15) as

$$\|y(k)\| \leq \beta + \alpha \beta^2 + K \frac{(T+1)^2}{8} < R, \quad k = 0, \dots, T+1, \tag{2.18}$$

and this concludes the proof. □

*Definition 2.2.* Let  $R > 0$  be a constant. Define the set  $D_R \subset \{1, \dots, T\} \times \mathbb{R}^d \times \mathbb{R}^d$  by the set containing all triplets  $(k, p, q)$  such that

$$k = 1, \dots, T, \quad (p, q) \in \mathbb{R}^{2d} : \|p\| \geq R, \quad 2\langle p, q \rangle + \|q\|^2 \leq 0. \tag{2.19}$$

By using the same selector technique as above, but employing an unrelated inequality, we now prove the second *a priori* bound result.

**LEMMA 2.3.** *Let  $R > 0$  be a constant such that*

$$\max \{ \|A\|, \|B\| \} < R, \tag{2.20}$$

*and let  $F : \{1, \dots, T\} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow CK(\mathbb{R}^d)$  be a set-valued map. If*

$$2\langle p, \phi \rangle + \|q\|^2 > 0, \quad \text{for every } (k, p, q) \in D_R \text{ and all } \phi \in F(k, p, q), \tag{2.21}$$

*then all solutions  $\bar{y}$  for the system of discrete inclusions (2.1) satisfy*

$$\|y(k)\| < R, \quad k = 0, \dots, T+1, \tag{2.22}$$

*for  $\lambda \in [0, 1]$ .*

*Proof.* Suppose that  $\bar{y}$  is a solution of (2.1). As in the previous proof, we can deduce that  $\bar{y}$  is a solution of (2.5) for some selector  $\hat{f}(k, p, q) \in F(k, p, q)$ .

Assume that the conclusion is not true. Then the function  $r(k) := \|y(k)\|^2 - R^2$  must have a nonnegative maximum in  $\{0, \dots, T + 1\}$ . From the assumption (2.20), this maximum must be achieved in  $\{1, \dots, T\}$ . Choose  $c \in \{1, \dots, T\}$  such that  $r(c) = \max\{r(k); k \in \{1, \dots, T\}\}$  and suppose that there is no  $k < c$  for which  $r(k) = r(c)$ . This choice of  $c$  implies that the conditions

$$\Delta r(c) \leq 0, \tag{2.23}$$

$$\Delta^2 r(c - 1) \leq 0 \tag{2.24}$$

must be satisfied simultaneously. Since

$$\begin{aligned} \Delta r(c) &= \langle y(c) + y(c + 1), \Delta y(c) \rangle = \langle 2y(c) + \Delta y(c), \Delta y(c) \rangle \\ &= 2\langle y(c), \Delta y(c) \rangle + \|\Delta y(c)\|^2 \end{aligned} \tag{2.25}$$

holds, we rewrite (2.23) as

$$2\langle y(c), \Delta y(c) \rangle + \|\Delta y(c)\|^2 \leq 0. \tag{2.26}$$

Similarly as in the proof of Lemma 2.1, we obtain with the help of the product rule

$$\Delta^2 r(c - 1) \geq \|\Delta y(c)\|^2 + 2\langle y(c), \hat{f}(c, y(c), \Delta y(c)) \rangle > 0, \tag{2.27}$$

which contradicts (2.24). Hence,  $\|y(k)\| < R$  for  $k = 0, \dots, T + 1$ . □

Lemma 2.3 is a natural extension to the lower and upper solution methods used in [2] for the case  $d = 1$ . If the right-hand side in (1.3) is a single-valued map, then we have the following corollary to Lemma 2.3 for systems of discrete BVPs.

**COROLLARY 2.4.** *Let  $R > 0$  be a constant. If*

$$2\langle p, F(k, p, q) \rangle + \|q\|^2 > 0, \quad \forall (k, p, q) \in D_R, \tag{2.28}$$

$$\max\{\|A\|, \|B\|\} < R, \tag{2.29}$$

*then all solutions  $\bar{y}$  of (2.5) satisfy (2.22) for  $\lambda \in [0, 1]$ .*

*Remark 2.5.* Note that if  $F(k, y(k), \Delta y(k)) = F(k, y(k))$ , then in place of (2.28), we would require only

$$\langle p, F(k, p) \rangle > 0, \quad \forall k = 1, \dots, T \text{ and all } p \in \mathbb{R}^d : \|p\| \geq R \tag{2.30}$$

to be satisfied.

The advantage of (2.19) rather than assuming (2.28) for, say, all  $q \in \mathbb{R}^d$  is highlighted in the following example.

*Example 2.6.* Consider the single, scalar-valued function  $f(k, p, q) = p^3 - q$ . For all  $q \in \mathbb{R}$ , we have that

$$2pf(k, p, q) + q^2 = 2p^4 - 2pq + q^2, \tag{2.31}$$

and for all  $|p| \geq R$ , we can find  $q \in \mathbb{R}$  such that (2.31) is negative. But on the reduced set  $D_R$ , the assumption (2.19) implies that  $2pq + q^2 \leq 0$  ( $-2pq \geq q^2$  equivalently), and thus

$$2pf(k, p, q) + q^2 = 2p^4 - 2pq + q^2 \geq 2p^4 + q^2 + q^2 > 0, \tag{2.32}$$

for all  $(p, q) \in \mathbb{R}^2$ ,  $|p| \geq R$ , such that (2.19) holds for any  $R > 0$ . Hence we can use Corollary 2.4 to prove the *a priori* bound for the discrete BVP,

$$\Delta^2 y(k - 1) = \lambda y^3(k) - \lambda \Delta y(k), \quad k = 1, \dots, T, \quad y(0) = \lambda A, \quad y(T + 1) = \lambda B, \tag{2.33}$$

where  $\lambda \in [0, 1]$ .

### 3. Operator formulation

In this section, we formulate the necessary operators to apply Theorem 1.2.

Solving (2.1) is equivalent to finding a vector  $\bar{y} = \{y(k)\}_{k=0}^{T+1} \in \mathbb{R}^{(T+2)d}$  which satisfies

$$y(k) \in \lambda \Phi(k) + \lambda \sum_{l=1}^T G(k, l) F(l, y(l), \Delta y(l)), \quad k = 0, \dots, T + 1, \tag{3.1}$$

where  $\lambda \Phi$  is the function defined by (2.7) and  $G : \{0, \dots, T + 1\} \times \{1, \dots, T\} \rightarrow \mathbb{R}^d$ , defined by (2.8), is the Green's function for the BVP

$$\Delta^2 y(k - 1) = 0, \quad k = 0, \dots, T, \quad y(0) = 0, \quad y(T + 1) = 0. \tag{3.2}$$

We suppose that  $F : \{1, \dots, T\} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow CK(\mathbb{R}^d)$  then we can define the operator  $\mathcal{F} : \mathbb{R}^{(T+2)d} \rightarrow CK(\mathbb{R}^{Td})$  by

$$\mathcal{F}(\bar{u}) := \{\bar{v} \in \mathbb{R}^{Td} : v(k) \in F(k, u(k), \Delta u(k)), k = 1, \dots, T\}, \tag{3.3}$$

and the operator  $\mathcal{T} : \mathbb{R}^{Td} \rightarrow \mathbb{R}^{(T+2)d}$  by

$$\mathcal{T}y(k) := \sum_{l=1}^T G(k, l)y(l), \quad k = 1, \dots, T. \tag{3.4}$$

The discreteness of topology implies that  $\mathcal{T}$  is continuous and linear, and thus, we have  $\mathcal{T} \circ \mathcal{F} : \mathbb{R}^{(T+2)d} \rightarrow CK(\mathbb{R}^{(T+2)d})$ . We can rewrite (3.1) as

$$\bar{y} \in \hat{\mathcal{F}}(\bar{y}), \tag{3.5}$$

where  $\widehat{\mathcal{G}} : \mathbb{R}^{(T+2)d} \rightarrow CK(\mathbb{R}^{(T+2)d})$  is defined by

$$\widehat{\mathcal{G}}(\bar{y}) := \lambda \mathcal{T} \circ \mathcal{F}(\bar{y}) + \lambda \Phi. \tag{3.6}$$

In order to use Theorem 1.2, we need to define a suitable open subset  $U$  of the Banach space  $\mathbb{R}^{(T+2)d}$  and the operator  $\mathcal{S} : \bar{U} \rightarrow CK(\mathbb{R}^{(T+2)d})$ . Introduce  $U \subset \mathbb{R}^{(T+2)d}$  by

$$U := \{\bar{u} \in \mathbb{R}^{(T+2)d} : \|u(k)\| < R, k = 0, \dots, T+1\}, \tag{3.7}$$

where  $R$  is the constant defined either in (2.4) or in Lemma 2.3. Next, we define the operator  $\mathcal{S} : \bar{U} \rightarrow CK(\mathbb{R}^{(T+2)d})$  by

$$\mathcal{S} = \widehat{\mathcal{G}}|_{\bar{U}}. \tag{3.8}$$

To satisfy the remaining assumptions of Theorem 1.2 on the operator  $\mathcal{S}$ , we need to prove that it is upper semicontinuous and compact.

LEMMA 3.1. *If  $F$  satisfies*

- (US)  *$F(k, p, q)$  is upper semicontinuous for all  $(p, q) \in \mathbb{R}^{2d}$ , for  $k = 1, \dots, T$  and the assumptions either of Lemma 2.1 or Lemma 2.3 hold, then the operator  $\mathcal{S}$  is upper semicontinuous and compact.*

*Proof.* Consider the sequences  $\{u_i\}_{i=1}^\infty$  and  $\{w_i\}_{i=1}^\infty$  such that  $w_i \in \mathcal{S}(u_i)$  and  $u_i \rightarrow u_0$  and  $w_i \rightarrow w_0$  as  $i \rightarrow \infty$  in  $\mathbb{R}^{(T+2)d}$ . To prove our assertion, we must show that  $w_0 \in \mathcal{S}(u_0)$ . For each  $i \in \mathbb{N}$ , there exists  $v_i \in \mathbb{R}^{Td}$  such that  $w_i = \lambda \mathcal{T} v_i + \lambda \Phi$  and  $v_i \in \mathcal{F}(u_i)$ . Since condition (US) holds and  $\bar{U}$  is a compact set, we can deduce from Theorem 1.3 that  $\mathcal{F}(\bar{U})$  is a compact set. This implies that there exists at least a subsequence  $\{v_{i_n}\}_{n=1}^\infty$  of  $\{v_i\}_{i=1}^\infty$  such that  $v_{i_n} \rightarrow v_0 \in \mathcal{F}(u_0)$ . Since  $\mathcal{T}$  is linear and continuous, we have that

$$w_i = \lambda \mathcal{T} v_i + \lambda \Phi \longrightarrow \lambda \mathcal{T} v_0 + \lambda \Phi, \tag{3.9}$$

and noting that  $w_i \rightarrow w_0$  in  $\mathbb{R}^{(T+2)d}$ , we can conclude that

$$w_0 \in \mathcal{S}(u_0). \tag{3.10}$$

This proves that  $\mathcal{S}$  is upper semicontinuous and we can use Theorem 1.3 to obtain the compactness of  $\mathcal{S}$ . □

#### 4. Existence results and examples

In this section, we combine the theory of Sections 2 and 3 to formulate existence results.

THEOREM 4.1. *If the set-valued map  $F : \{1, \dots, T\} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow CK(\mathbb{R}^d)$  satisfies (US) and the assumptions of Lemma 2.1 hold, then the system of discrete inclusion boundary value problems (1.3) has a solution.*

*Proof.* We showed in the previous section that the problem of existence of a solution of (1.3) (where  $F$  satisfies the assumptions of Lemma 2.1) is equivalent to the problem of existence of a fixed point of  $\mathcal{S}(\bar{y})$ , where  $\mathcal{S} : \bar{U} \rightarrow CK(\mathbb{R}^{(T+2)d})$  is defined in (3.8) and  $U$  is

defined in (3.7). Since  $\mathcal{S}$  is compact and upper semicontinuous (cf. Lemma 3.1), we are ready to use Theorem 1.2, and thanks to the conclusion of Lemma 2.1, we can exclude the possibility (A2) there. Therefore the operator  $\mathcal{S}$  has a fixed point and the problem (1.3) has a solution.  $\square$

We illustrate the above result on the following example with  $n = 1$ .

*Example 4.2.* Let  $\tilde{F} : \{1, \dots, T\} \times \mathbb{R} \rightarrow CK(\mathbb{R})$  be a set-valued map defined by

$$\tilde{F}(k, p) := \bigcup_{\epsilon \in [-1; 1]} (k \pm \epsilon)p^5. \tag{4.1}$$

Consider the BVP

$$\Delta^2 y(k-1) \in \tilde{F}(k, y(k)), \quad k = 1, \dots, T, \quad y(0) = A, \quad y(T+1) = B, \tag{4.2}$$

where  $A, B \in \mathbb{R}$ . Since  $k \in \{1, \dots, T\}$  and  $|\epsilon| \leq 1$ , the inequality  $(k \pm \epsilon)p^5 \leq (k \pm \epsilon)(p^6 + 1)$  holds and thus for every selector  $\tilde{f}(k, p) \in \tilde{F}(k, p)$ , we have that

$$\begin{aligned} |\tilde{f}(k, p)| &\leq (k \pm \epsilon)(p^6 + 1) = p(k \pm \epsilon)p^5 + t \pm \epsilon \\ &\leq p\tilde{f}(k, p) + T + 1, \end{aligned} \tag{4.3}$$

and the inequality (2.2) is satisfied with  $\alpha = 1/2$  and  $K = T + 1$ . The set-valued map  $\tilde{F}$  satisfies (US) and thus we can use Theorem 4.1 to prove that the problem (4.2) has a solution.

As a natural corollary to Theorem 4.1, we have the following result.

**COROLLARY 4.3.** *Let  $K \geq 0$  be a constant. If the set-valued map  $F : \{1, \dots, T\} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow CK(\mathbb{R}^d)$  satisfies (US) and*

$$\|\phi\| \leq K, \quad \text{for } k = 1, \dots, T \text{ and all } (p, q) \in \mathbb{R}^{2d}, \phi \in F(k, p, q), \tag{4.4}$$

*then the system of discrete inclusion boundary value problems (1.3) has a solution.*

*Proof.* See that this is a special case of Theorem 4.1 with  $\alpha = 0$ .  $\square$

We now prove an existence result for the conditions from Lemma 2.3.

**THEOREM 4.4.** *If the set-valued map  $F : \{1, \dots, T\} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow CK(\mathbb{R}^d)$  satisfies (US) and the assumptions of Lemma 2.3 hold, then the system of discrete inclusion boundary value problems (1.3) has a solution.*

*Proof.* The proof is identical to that of Theorem 4.1. The only difference consists of the different definition of the constant  $R$  when defining the set  $\bar{U}$ , see Lemma 2.3.  $\square$

We illustrate the above result with an example in two dimensions.

*Example 4.5.* Let  $\hat{F}: \{1, \dots, T\} \times \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow CK(\mathbb{R}^2)$  be a set-valued map defined by

$$\hat{F}(k, p, q) := \bigcup_{\epsilon \in (0;1]} \left( \begin{matrix} \epsilon k p_1 - q_1 \\ p_2^3(1 + q_1^2) - p_2 - q_2 \end{matrix} \right), \tag{4.5}$$

where  $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$  and  $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ . Consider the BVP

$$\Delta^2 y(k-1) \in \hat{F}(k, y(k), \Delta y(k)), \quad k = 1, \dots, T, \quad y(0) = A, \quad y(T+1) = B, \tag{4.6}$$

where  $A, B \in \mathbb{R}^2$ . Let  $\hat{f}(k, p, q) \in \hat{F}(k, p, q)$  be an arbitrary selector for all  $k = 1, \dots, T$ ,  $q, p \in \mathbb{R}^d$ ,  $\|p\| \geq R$ , where  $R > 1$  and  $p, q$  are such that  $2\langle p, q \rangle + \|q\|^2 \leq 0$ , for example,

$$2p_1q_1 + 2p_2q_2 + q_1^2 + q_2^2 \leq 0. \tag{4.7}$$

We make the following calculation:

$$\begin{aligned} 2\langle p, \hat{f}(k, p, q) \rangle + \|q\|^2 &= 2 \left\langle \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \begin{pmatrix} k p_1 - q_1 \\ p_2^3(1 + q_1^2) - p_2 - q_2 \end{pmatrix} \right\rangle + q_1^2 + q_2^2 \\ &= 2\epsilon k p_1^2 + 2p_2^4(1 + q_1^2) - 2p_2^2 - 2p_1q_1 - 2p_2q_2 + q_1^2 + q_2^2. \end{aligned} \tag{4.8}$$

Using (4.7), we can provide the estimation

$$2\langle p, \hat{f}(k, p, q) \rangle + \|q\|^2 \geq 2\epsilon k p_1^2 + 2p_2^4(1 + q_1^2) - p_2^2 + 2q_1^2 + 2q_2^2 > 0, \tag{4.9}$$

since either  $\epsilon p_1^2 > 0$  or  $p_2^4(1 + q_1^2) - p_2^2 > 0$ . The set-valued map  $\hat{F}$  satisfies also the condition (US), and thus we can use Theorem 4.4 to prove that the problem (4.6) has a solution.

### Acknowledgments

The research of the first author was supported by the Grant Agency of Czech Republic, no. 201/03/0671. The research of the second author was supported by the Australian Research Council’s Discovery Projects (DP0450752). The research of both authors was conducted at the University of New South Wales (UNSW), Sydney, Australia.

### References

- [1] R. P. Agarwal, *Difference Equations and Inequalities. Theory, Methods, and Applications*, 2nd ed., Monographs and Textbooks in Pure and Applied Mathematics, vol. 228, Marcel Dekker, New York, 2000.
- [2] R. P. Agarwal, D. O’Regan, and V. Lakshmikantham, *Discrete second order inclusions*, J. Differ. Equations Appl. **9** (2003), no. 10, 879–885.
- [3] J.-P. Aubin and A. Cellina, *Differential Inclusions. Set-Valued Maps and Viability Theory*, Grundlehren der Mathematischen Wissenschaften, vol. 264, Springer-Verlag, Berlin, 1984.
- [4] C. Berge, *Espaces Topologiques: Fonctions Multivoques*, Collection Universitaire de Mathématiques, vol. 3, Dunod, Paris, 1959.
- [5] P. Diamond and P. Kloeden, *Metric Spaces of Fuzzy Sets. Theory and Applications*, World Scientific, New Jersey, 1994.

- [6] L. Erbe, R. Ma, and C. C. Tisdell, *On two point boundary value problems for second order differential inclusions*, to appear in *Dynam. Systems Appl.*
- [7] W. G. Kelley and A. C. Peterson, *Difference Equations. An Introduction with Applications*, 2nd ed., Harcourt/Academic Press, California, 2001.
- [8] G. V. Smirnov, *Introduction to the Theory of Differential Inclusions*, Graduate Studies in Mathematics, vol. 41, American Mathematical Society, Rhode Island, 2002.

Petr Stehlík: Department of Mathematics, Faculty of Applied Sciences, University of West Bohemia, Univerzitní 22, 306 14 Plzeň, Czech Republic

*E-mail address:* pstehlik@kma.zcu.cz

*Current address:* School of Mathematics, The University of New South Wales, Sydney NSW 2052, Australia

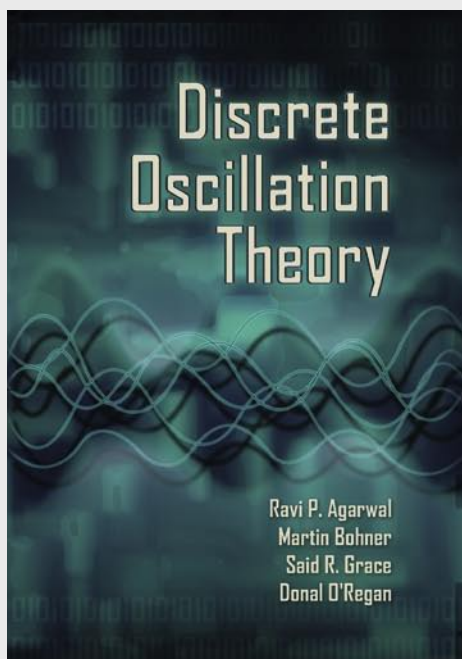
Christopher C. Tisdell: School of Mathematics, The University of New South Wales, Sydney NSW 2052, Australia

*E-mail address:* cct@maths.unsw.edu.au

CMIA Book Series, Volume 1, ISBN: 977-5945-19-4

## DISCRETE OSCILLATION THEORY

Ravi P. Agarwal, Martin Bohner, Said R. Grace, and Donal O'Regan



This book is devoted to a rapidly developing branch of the qualitative theory of difference equations with or without delays. It presents the theory of oscillation of difference equations, exhibiting classical as well as very recent results in that area. While there are several books on difference equations and also on oscillation theory for ordinary differential equations, there is until now no book devoted solely to oscillation theory for difference equations. This book is filling the gap, and it can easily be used as an encyclopedia and reference tool for discrete oscillation theory.

In nine chapters, the book covers a wide range of subjects, including oscillation theory for second-order linear difference equations, systems of difference equations, half-linear difference equations, nonlinear difference equations, neutral difference equations, delay difference equations, and differential equations with piecewise constant arguments. This book summarizes almost 300 recent research papers and hence covers all aspects of discrete oscillation theory that have been discussed in recent journal articles. The presented

theory is illustrated with 121 examples throughout the book. Each chapter concludes with a section that is devoted to notes and bibliographical and historical remarks.

The book is addressed to a wide audience of specialists such as mathematicians, engineers, biologists, and physicists. Besides serving as a reference tool for researchers in difference equations, this book can also be easily used as a textbook for undergraduate or graduate classes. It is written at a level easy to understand for college students who have had courses in calculus.

For more information and online orders please visit <http://www.hindawi.com/books/cmia/volume-1>  
For any inquires on how to order this title please contact [books.orders@hindawi.com](mailto:books.orders@hindawi.com)

"Contemporary Mathematics and Its Applications" is a book series of monographs, textbooks, and edited volumes in all areas of pure and applied mathematics. Authors and/or editors should send their proposals to the Series Editors directly. For general information about the series, please contact [cmia.ed@hindawi.com](mailto:cmia.ed@hindawi.com).

# 6th AIMS INTERNATIONAL CONFERENCE ON

# Dynamical Systems, Differential Equations & Applications 2006

Sunday-Wednesday June 25 - 28

University of Poitiers  
Poitiers, France

COMMITTEES • INVITED PLENARY SPEAKERS

**Alberto Bressan**  
(USA)

**Odo Diekmann**  
(Netherlands)

**Pierre-Louis Lions**  
(France)

**Alexander Melke**  
(Germany)

**Masayasu Mimura**  
(Japan)

**Peter Polacik**  
(USA)

**Patrizia Pucci**  
(Italy)

**Björn Sandstede**  
(UK)

**Lan Wen**  
(China)

**Organizing Committee:**  
Alain Miranville (Chair)  
Laurence Cherfils  
Hassan Emamirad  
Morgan Pierre  
Philippe Rogeon  
Arnaud Rougirel

**Scientific Committee:**  
Shouchuan Hu (Chair)  
Antonio Ambrosetti  
Jerry Bona  
Michel Chipot  
Tom Hou  
Tatsien Li  
Carlangelo Liverani  
Hiroshi Matano  
Alain Miranville  
Wei-Ming Ni  
Juergen Sprekels  
Roger Temam

## Program and Topics

- **Topics:** The conference will cover all the major research areas in analysis and dynamics with focuses on applications, modeling and computations.
- **Format:** There will be one-hour plenary talks, and 30-minute special session and contributed session talks. The conference will be diversified in subject so as to attract all researchers in relevant fields; while it is concentrated enough in each chosen subfield so as to sustain a large number of strong special sessions.
- **Deadlines:** March 1, 2006, for registration and abstracts.

## Proceedings

- A proceedings will be published as a special edition of DCDS. All papers will be refereed.

## Funding

- Funding from various sources including NSF may be available to qualified graduate students and young researchers.

## Contact Information

- Detailed program and updates will be posted at <http://aimSciences.org>. For updated information, contact
  - **Alain Miranville**, Chair of Organizing Committee (France)  
Tel: (33) (0)5 49 49 68 91, Fax: (33) (0)5 49 49 69 01  
[Alain.Miranville@math.univ-poitiers.fr](mailto:Alain.Miranville@math.univ-poitiers.fr)
  - **Xin Lu**, Conference Coordinator (USA)  
Tel: 910-962-3673, Fax: 910-962-7107  
[lux@uncw.edu](mailto:lux@uncw.edu)



Organized by:

- American Institute of Mathematical Sciences (AIMS)
- Department of Mathematics, University of Poitiers.