On the Solvability of Two-point, Second-order Boundary Value Problems

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Abstract

We gain solvability to a system of nonlinear, second-order ordinary differential equations subject to a range of boundary conditions. The ideas involve differential inequalities and fixed point methods. In particular, maximum principles are not employed.

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1 Introduction

This paper considers the existence of solutions to the nonlinear boundary value problem (BVP)

$$x'' = f(t, x, x'), \qquad \text{for} \quad t \in (0, T), \tag{1}$$

$$x \in \beta_0, \tag{2}$$

where $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous, nonlinear function and (2) represents any set of linear, homogeneous two-point boundary conditions, including:

$$\begin{aligned} x(0) = \;& 0 \; = x(T), && \text{Dirichlet conditions;} && (3)\\ x'(0) = \;& 0 \; = x(T), && \text{left focal conditions;} && (4)\\ x(0) = \;& 0 \; = x'(T), && \text{right focal conditions;} && (5)\\ x'(0) = \;& 0 \; = x'(T), && \text{Neumann conditions;} && (6)\\ x(0) = x(T), \quad & x'(0) = x'(T), && \text{periodic conditions.} && (7) \end{aligned}$$

A solution to (1), (2) is a continuously twice-differentiable function $x : [0, T] \to \mathbb{R}^n$, i.e., $x \in C^2([0, T]; \mathbb{R}^n)$, that satisfies both (1) and (2).

The wide applications of BVPs to physics, engineering and science are well known [3, Chap.1], [1, Chap.1], and these applications naturally motivate a deeper theoretical study of the subject. We note that maximum principle techniques, including the much celebrated use of upper and lower solutions and their generalization to systems of equations, have dominated the field of solvability theory for BVPs. This work pursues an alternate approach. Instead of using maximum principles, we employ a general growth condition motivated by the classical and influential work of Hartman [8].

Using novel differential inequalities and standard fixed-point methods, we establish an abstract existence result for (1), (2) in Section 2. Sections 3 and 4 then demonstrate some applications of the aforementioned abstract result to a variety of BVPs. As for notation, if $y, z \in \mathbb{R}^n$, then $\langle y, z \rangle$ denotes their usual inner product and $\|z\|$ denotes the Euclidean norm of z. We adopt the standard norm for elements v of $C^1([0, T]; \mathbb{R}^n)$, namely

$$\|v\|_1 := \max\{\max_{t \in [0,T]} \|v(t)\|, \max_{t \in [0,T]} \|v'(t)\|\}\,.$$

For more on solvability to BVPs, including modern and classical approaches, see [1]-[11] and [13]-[18].

2 An Abstract Solvability Result

Our main abstract existence result is

Theorem 2.1 *Let* α, r, K *and* N *be non-negative constants and let* f *be continuous. Suppose: that the linear BVP*

$$x'' - rx = 0 \quad \textit{for} \quad t \in (0,T), \qquad x \in \beta_0 \tag{8}$$

has only the zero solution; that

$$\| \, f(t,p,q) - rp \, \| \le 2\alpha \left[\, \langle p, f(t,p,q) \rangle + \|q\|^2 \, \right] + K \tag{9}$$

for all $(t,p,q) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$*; and that* (2) *implies*

$$| \, \langle x(T), x'(T) \rangle - \langle x(0), x'(0) \rangle \, | \le N. \tag{10}$$

Then (1), (2) *has at least one solution.*

Proof Consider the following BVP, which is equivalent to (1), (2):

$$x'' - rx = f(t,x,x') - rx, \qquad t \in (0,T), \tag{11}$$

$$x \in \beta_0 \, . \tag{12}$$

Since (8) has only the zero solution, there exists a unique, continuously differentiable Green's function $G : [0,T] \times [0,T] \to \mathbb{R}$ such that (11), (12) may be equivalently reformulated as the integral equation

$$x(t) = \int_0^T G(t,s) \, [f(s,x(s),x'(s)) - rx(s)] \;\, ds, \quad t \in [0,T]. \tag{13}$$

We therefore define $H : C^1([0,T];\mathbb{R}^n) \to C^1([0,T];\mathbb{R}^n)$ by

$$(Hx)\,(t) := \int_0^T G(t,s) \, [f(s,x(s),x'(s)) - rx(s)] \;\, ds, \quad t \in [0,T] \tag{14}$$

and consider the family of equations

$$u = \lambda H u, \quad \lambda \in (0,1). \tag{15}$$

Since Hx actually belongs to $C^2([0,T];\mathbb{R}^n)$ for each $x \in C^1([0,T];\mathbb{R}^n)$, it follows from the compact embedding of $C^2([0,T];\mathbb{R}^n)$ into $C^1([0,T];\mathbb{R}^n)$ that H is a compact map.

We will apply the Schaefer Fixed Point Theorem [12, Theorem 4.4.10] to prove that H has at least one fixed point in $C^1([0,T];\mathbb{R}^n)$. Since $H : C^1([0,T];\mathbb{R}^n) \to C^1([0,T];\mathbb{R}^n)$ is compact, it remains to verify that all solutions to (15) are bounded independently of λ. With this in mind, suppose that u satisfies (15) and define

$$G_0 := \max_{(t,s) \in [0,T] \times [0,T]} |G(t,s)|.$$

We then have, for each $t \in [0,T]$ and $\lambda \in [0,1]$,

$$\begin{aligned}
\|u(t)\| &\le \max_{(t,s)\in[0,T]\times[0,T]} |G(t,s)| \int_0^T \lambda \|f(s,u(s),u'(s)) - ru(s)\| \, ds \\
&\le G_0 \int_0^T \lambda \left(2\alpha \left[\langle u(s), f(s,u(s),u'(s))\rangle + \|u'(s)\|^2 \right] + K \right) \, ds \\
&\le G_0 \int_0^T 2\alpha \left[\langle u(s), \lambda f(s,u(s),u'(s)) + (1-\lambda) r u(s)\rangle + \|u'(s)\|^2 \right] + \lambda K \, ds \\
&\le G_0 \int_0^T 2\alpha \left[\, \langle u(s), u''(s)\rangle + \|u'(s)\|^2 \, \right] + K \, ds \\
&= G_0 \int_0^T \alpha \frac{d^2}{ds^2} \left[\, \|u(s)\|^2 \, \right] + K \, ds \\
&= G_0 \left[2\alpha(\langle u(T), u'(T)\rangle - \langle u(0), u'(0)\rangle) + KT \right] \\
&\le G_0 [2\alpha N + KT], \quad \text{from (10).}
\end{aligned}$$

A similar calculation yields an estimate on u': differentiating both sides of the integral equation and taking norms yields, for each $t \in [0,T]$,

$$\|u'(t)\| \le G_1 [2\alpha N + KT] \, , \quad \text{where} \quad G_1 := \max_{(t,s)\in[0,T]\times[0,T]} \left| \frac{\partial G}{\partial t}(t,s) \right| .$$

So the bounds on possible solutions to (15) are independent of λ and applying Schaefer's Fixed Point Theorem completes the proof. □

3 Applications of the Abstract Theorem

We now illustrate how Theorem 2.1 may be applied to a range of BVPs.

Theorem 3.1 *Let α and K be non-negative constants and let f be continuous. If*

$$\|f(t,p,q)\| \le 2\alpha \left[\langle p, f(t,p,q)\rangle + \|q\|^2 \right] + K, \quad \forall (t,p,q) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n, \quad (16)$$

then each of the following BVPs has at least one solution: (1), (3)*;* (1), (4)*;* (1), (5)*.*

Proof The linear differential equation $x'' = 0$ subject to (3), (4) or (5) has only the zero solution. Thus we may choose $r = 0$ in (8), and (9) then follows directly from (16). Obviously, each of (3), (4) and (5) imply that (10) holds for $N = 0$. The conditions of Theorem 2.1 are therefore satisfied and the solvability result follows. □

Theorem 3.2 *Let α and K be non-negative constants and let f be continuous. If*

$$\|f(t,p,q)-p\| \le 2\alpha\left[\langle p, f(t,p,q)\rangle + \|q\|^2\right] + K, \quad \forall (t,p,q) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^n, \tag{17}$$

then each of the following BVPs has at least one solution: (1), (6)*;* (1), (7)*.*

Proof Firstly, the linear differential equation $x'' - x = 0$ subject to (6) or (7) has only the zero solution. Thus we may choose $r = 1$ in (8). Secondly, if (17) holds, then (9) holds with $r = 1$. Since (6) and (7) imply that (10) holds with $N = 0$, the conditions of Theorem 2.1 are satisfied and the solvability result follows. □

The solvability of the BVP (1), (3) furnished by Theorem 3.1 is contained in Hartman [8]. We have included the above case for completeness. The rest of the existence results in this section are new.

If f does not depend on x' then (16) and (17) may be replaced, respectively, by the simpler inequalities

$$\|f(t,p)\| \le 2\alpha\langle p, f(t,p)\rangle + K, \quad \forall (t,p) \in [0,T]\times\mathbb{R}^n;$$
$$\|f(t,p)-p\| \le 2\alpha\langle p, f(t,p)\rangle + K, \quad \forall (t,p) \in [0,T]\times\mathbb{R}^n;$$

with new existence results holding for the case $f = f(t,x)$ subject to (2). We omit the statements of these new theorems for brevity.

4 Examples

In this final section, we consider some simple classes of examples to which our new theorems are applicable.

The first example deals with scalar BVPs.

Example 1 Consider (1), (7) with f being scalar-valued and given by

$$f(t,p,q) = q^2p^3 + t + p, \quad t \in [0,1].$$

For the above f we claim that (9) holds with $r = 1$. We see that

$$|f(t,p,q)-p| \le q^2|p|^3 + 1, \quad \forall (t,p,q) \in [0,1]\times\mathbb{R}^2$$

and, for α and K to be chosen below

$$\begin{aligned} 2\alpha[pf(t,p,q)+q^2]+K &= 2\alpha[q^2(p^4+1)+p^2+pt]+K \\ &\ge q^2|p|^3+p^2+pt+K, \quad \text{for } \alpha = 1/2 \\ &\ge q^2|p|^3+1, \quad \text{for } K = 100. \end{aligned}$$

Thus (9) holds and the existence of at least one solution follows from Theorem 2.1. □

Our second example deals with systems of BVPs.

Example 2 Consider (1), (4) with $n = 2$, f not depending on x', and f is given by

$$\begin{aligned} f(t,p) &= (h(t,y,z), j(t,y,z)), \quad t \in [0,1], \\ &= ((t+1)y^3 + ye^{-z^2} + 1, (t+1)z^3 + ze^{-y^2}). \end{aligned} \tag{18}$$

We claim that the above f satisfies the conditions of Theorem 3.1. Note that for all $(t,p) \in [0,1] \times \mathbb{R}^2$ we have

$$\begin{aligned} \|f(t,p)\| &\leq |h(t,y,z)| + |j(t,y,z)| \\ &\leq 2|y|^3 + |y|e^{-z^2} + 2|z|^3 + |z|e^{-y^2} + 1. \end{aligned}$$

Below, we will need the following simple inequalities:

$$\begin{aligned} w^4 \geq |w|^3 - 1, \quad w^4 + w \geq |w|^3 - 10, \quad \text{for all } w \in \mathbb{R}, \\ b^2e^{-a^2} \geq |b|e^{-a^2} - 1, \quad \text{for all } (a,b) \in \mathbb{R}^2. \end{aligned}$$

For $\alpha \geq 0$ and $K \geq 0$ to be chosen below, consider for $(t,p) \in [0,1] \times \mathbb{R}^2$,

$$\begin{aligned} & 2\alpha\langle p, f(t,p)\rangle + K \\ \geq\ & 2\alpha\left[y^4 + y + y^2e^{-z^2} + z^4 + z^2e^{-y^2}\right] + K \\ \geq\ & 2\alpha\left[|y|^3 - 10 + |y|e^{-z^2} - 1 + |z|^3 - 1 + |z|e^{-y^2} - 1\right] + K \\ \geq\ & 2|y|^3 + |y|e^{-z^2} + 2|z|^3 + |z|e^{-y^2} + 1 \geq \|f(t,p)\| \\ & \text{choosing } \alpha = 1,\ K = 27. \end{aligned}$$

Thus f satisfies the conditions of Theorem 3.1 for the choices $\alpha = 1$ and $K = 27$. □

References

[1] Agarwal, Ravi P. Boundary value problems for higher order differential equations. World Scientific Publishing Co., Inc., Teaneck, NJ, 1986.

[2] Amster, P.; Rogers, C.; Tisdell, C. C. Existence of solutions to boundary value problems for dynamic systems on time scales. J. Math. Anal. Appl. 308 (2005), no. 2, 565–577.

[3] Ascher, Uri M.; Mattheij, Robert M. M.; Russell, Robert D. Numerical solution of boundary value problems for ordinary differential equations. Corrected reprint of the 1988 original. Classics in Applied Mathematics, 13. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1995.

[4] Handbook of differential equations. Ordinary differential equations. Vol. I. Edited by A. Canada, P. Drábek and A. Fonda. Elsevier/North-Holland, Amsterdam, 2004.

[5] Handbook of differential equations: ordinary differential equations. Vol. II. Edited by A. Canada, P. Drábek and A. Fonda. Elsevier B. V., Amsterdam, 2005.

[6] Erbe, L.; Peterson, A.; Tisdell, C. C. Existence of solutions to second-order BVPs on time scales. Appl. Anal. 84 (2005), no. 10, 1069–1078.

[7] Granas, Andrzej; Guenther, Ronald; Lee, John. Nonlinear boundary value problems for ordinary differential equations. Dissertationes Math. (Rozprawy Mat.) 244 (1985), 128 pp.

[8] Hartman, Philip. On boundary value problems for systems of ordinary, nonlinear, second order differential equations. Trans. Amer. Math. Soc. 96 (1960), 493–509.

[9] Henderson, Johnny; Peterson, Allan; Tisdell, Christopher C. On the existence and uniqueness of solutions to boundary value problems on time scales. Adv. Difference Equ. 2004, no. 2, 93–109.

[10] Henderson, Johnny; Tisdell, Christopher C. Topological transversality and boundary value problems on time scales. J. Math. Anal. Appl. 289 (2004), no. 1, 110–125.

[11] Henderson, Johnny; Tisdell, C. C. Dynamic boundary value problems of the second-order: Bernstein-Nagumo conditions and solvability. Nonlinear Anal. (in press) doi:10.1016/j.na.2006.07.023

[12] Lloyd, N. G. Degree theory. Cambridge Tracts in Mathematics, No. 73. Cambridge University Press, Cambridge-New York-Melbourne, 1978.

[13] Peterson, A. C.; Raffoul, Y. N.; Tisdell, C. C. Three point boundary value problems on time scales. J. Difference Equ. Appl. 10 (2004), no. 9, 843–849.

[14] Rachunkova, I.; Stanek, S.; Tvrdy, M. Singularities and Laplacians in Boundary Value Problems for Nonlinear Ordinary Differential Equations. Handbook of Differential Equations. Ordinary Differential Equations, Ed. by A. Canada, P. Drábek, A. Fonda, Vol. 3., pp. 605-721, Elsevier 2006.

[15] Tisdell, C. C. Existence of solutions to first-order periodic boundary value problems. J. Math. Anal. Appl. (in press) doi:10.1016/j.jmaa.2005.11.047

[16] Tisdell, C. C. On the solvability of nonlinear first-order boundary-value problems. Electron. J. Differential Equations. Vol. 2006(2006), No. 80, pp. 1-8.

[17] Tisdell, Christopher C.; Tan, Lit Hau. On vector boundary value problems without growth restrictions. JIPAM. J. Inequal. Pure Appl. Math. 6 (2005), no. 5, Article 137, 10 pp. (electronic).

[18] Tisdell, C. C.; Thompson, H. B. On the existence of solutions to boundary value problems on time scales. Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 12 (2005), no. 5, 595–606.