MATH2019 Problems

1. Partial Differentiation

1 Verify that $w_{xy} = w_{yx}$ if **a.** $w = \ln(2x+3y)$ **b.** $w = e^x \sinh y + \cos(2x-3y)$.

2 Find $\dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial y^2}, \dfrac{\partial^2 f}{\partial y \partial x}$.

a. $f(x,y) = \ln(x^2+y^2)$;
b. $f(x,y) = x^2 y + \cos y + y \sin x$;
c. $f(x,y) = \tan^{-1} \dfrac{y}{x}$.

3 Show that the following functions satisfy the Laplace equation: $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} = 0$.

a. $f(x,y,z) = 2z^3 - 3(x^2+y^2)z$;
b. $f(x,y) = \ln \sqrt{x^2+y^2}$;
c. $f(x,y,z) = e^{3x+4y} \cos 5z$.

4 Show that the following functions are solutions of the wave equation

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}.$$

a. $w = \cos(2x+2ct)$;
b. $w = \ln(3x+3ct)$.

5 Use the chain rule to express $\dfrac{df}{dt}$ in terms of t. Then evaluate $\dfrac{df}{dt}$ at the given value of t

a. $f(x,y,z) = \ln(x+y+z),\ x = \cos^2 t,\ y = \sin^2 t,\ z = t;\ t = \pi$;
b. $f(x,y) = x^2 + y^2,\ x = \cos t,\ y = \sin t;\ t = \pi$.

6 If $u = x^2 + e^{y^2},\ x = \sin 2t$, and $y = \cos t^2$ find $\dfrac{du}{dt}$.

7 Find $\dfrac{\partial z}{\partial v}$ if $z = x^2 + 2xy,\ x = u\cos v,\ y = u \sin v$. (Express your answer in terms of x and y.)

8 Find $\dfrac{\partial z}{\partial u}$ when $u = 0,\ v = 1$ if

$$z = \sin xy + x \sin y,\ x = u^2 + v^2,\ y = uv.$$

9 Find $\dfrac{\partial w}{\partial v}$ when $u = 0,\ v = 0$ if

$$w = (x^2+y-2)^4 + (x-y+2)^3,\ x = u - 2v + 1,\ y = 2u + v - 2.$$

10 Find $\dfrac{\partial w}{\partial x}$ at the point $(x,y,z) = (1,1,1)$ if

$$w = \cos uv,\ u = xyz,\ v = \frac{\pi}{(4(x^2+y^2))}.$$

11 If $z = f(t)$ and $t = \dfrac{(x+y)}{xy}$, show that

$$x^2 \frac{\partial z}{\partial x} = y^2 \frac{\partial z}{\partial y}.$$

12 If a and b are constants, $w = f(u)$, and $u = ax + by$, show that

$$a\frac{\partial w}{\partial y} = b\frac{\partial w}{\partial x}.$$

13 If $w = f(u, v)$, $u = x + y$, and $v = x - y$, show that

$$\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} = \left(\frac{\partial f}{\partial u}\right)^2 - \left(\frac{\partial f}{\partial v}\right)^2.$$

14 If we substitute polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ in a continuous function $w = f(x, y)$ that has continuous partial derivatives, show that

$$\frac{\partial w}{\partial r} = f_x \cos\theta + f_y \sin\theta,$$

The Taylor series expansion of function $f(x, y)$ of two independent variables about a point (a, b) is

$$f(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) +$$

$$\frac{1}{2!}\left\{\frac{\partial^2 f}{\partial x^2}(a,b)(x-a)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(a,b)(x-a)(y-b) + \frac{\partial^2 f}{\partial y^2}(a,b)(y-b)^2\right\} + \cdots$$

15 Determine the Taylor series expansion of $f(x) = \sin x$ about
a. $x = 0$;
b. $x = \pi$, including the first two non-zero terms in each case. Sketch $f(x)$ and the two truncated expansions for $0 \le x \le \pi$.

16 Determine the Taylor series expansion of $f(x, y) = x^2 y$ about $(1, 2)$, including terms to 27th order.

17 Using a Taylor series in two variables, show that for small x and y we may make the following approximations
a. $e^x \sin y \sim y + xy$;
b. $e^x \ln(1 + y) \sim y + xy - \dfrac{y^2}{2}$.

18 Expand $\cos(2x - y)$ in a Taylor series in two variables, including quadratic terms, about
a. $(0, 0)$;
b. $(0, -\dfrac{\pi}{2})$.

19 Determine the Taylor expansion of $e^{x+y}\cos y$ about the point $(1, 0)$, up to and including quadratic terms.

20 Expand $\cos(2x - y)$ about $\left(\dfrac{\pi}{4}, \dfrac{\pi}{4}\right)$ up to and including second order terms using Taylor's series for functions of two variables.

21 Expand $\ln(x^2 + y^2)$ about $(1, 0)$ up to and including second order terms, using Taylor series for functions of two variables. Then use your result to find an approximate value for $\ln(1.1^2 + 0.1^2)$.

22 Calculate the Taylor expansion up to and including second order terms of the function

$$z = F(x, y) = e^{-x}\sin y$$

about the point $\left(2, \frac{\pi}{2}\right)$. Use your result to estimate $F\left(1.92, \frac{\pi}{2}\right)$.

23 Using an appropriate Taylor series approximtion, find an approximate value for $\sqrt{(1.02)^3 + (1.97)^3}$.

24 Suppose T is to be found from the formula $T = x\cosh y$, where x and y are found to be 2 and $\ln 2$ with maximum possible errors of $|dx| = 0.04$ and $|dy| = 0.02$. Estimate the maximum possible error in the computed value of T.

25 If $r = 5.0$ cm and $h = 12.0$ cm to the nearest millimetre, what should we expect the maximum percentage error in calculating $V = \pi r^2 h$ to be?

26 When an x-ohm and a y-ohm resistor are in parallel, the resistance R they produce will be calculated from the formula $\frac{1}{R} = \frac{1}{x} + \frac{1}{y}$. By what percentage will R change if x increases from 20 to 20.1 ohms and y decreases from 25 to 24.9 ohms?

27 The specific gravity δ of a solid heavier than water is given by

$$\delta = \frac{W}{W - W_1}$$

where W and W_1 are its weight in air and water respectively. W and W_1 are observed to by 17.2 and 9.7 gm. Find the maximum possible error in the calculated value of δ due to an error of 0.05 gm in each observation.

28 The pressure P, volume V and temperature T of a gas are related by the formula

$$PV = RT$$

where R is a constant. If V is increased by 10% and T decreased by 6%, find the percentage change in the pressure.

29 When two resistances r_1 and r_2 are connected in parallel, the total resistance R (measured in ohms) is given by:

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}.$$

If $r_1 = 6 \pm 0.1$ ohms and $r_2 = 9 \pm 0.03$ ohms

a. Calculate R;

b. Show that $\dfrac{\partial R}{\partial r_1} = \dfrac{R^2}{r_1^2}$;

c. Estimate the maximum possible error in the calculated value of R.

30 Show that

a. $\dfrac{d}{dx}\displaystyle\int_1^x t^2 dt = x^2$;

b. $\dfrac{d}{dt}\displaystyle\int_{t^3}^2 \ln(1 + x^2) dx = -3t^2 \ln(1 + t^6)$;

c. $\dfrac{d}{du}\displaystyle\int_0^{\pi/(2u)} u \sin(ux) dx = 0$.

31 Given that

$$\int_0^\infty e^{-ax} \sin(kx) dx = \frac{k}{a^2 + k^2},$$

deduce results for

a. $\int_0^\infty xe^{-ax} \sin(kx) dx$;

b. $\int_0^\infty xe^{-ax} \cos(kx) dx$.

32 Evaluate

$$\int_0^\infty (\alpha^2 + x^2)^{-1} dx,$$

and hence use Leibniz' theorem to deduce that

$$\int_0^\infty \frac{dx}{(\alpha^2 + x^2)^2} = \frac{\pi}{4\alpha^3}.$$

33 Given that

$$\int_0^\pi \frac{1}{\alpha - \cos\theta} d\theta = \frac{\pi}{\sqrt{\alpha^2 - 1}}, \qquad \text{for} \qquad \alpha > 1.$$

Use this result and Leibniz's Rule (for differentiating integrals) to evaluate

$$\int_0^\pi \frac{1}{(\alpha - \cos\theta)^2} d\theta \quad \text{for} \quad \alpha > 1.$$

34 Let $I(t) = \int_0^\pi \cos\, tx\, dx$. Show by simple integration that

$$I(t) = \frac{\sin t\pi}{t}$$

and then by differentiation that

$$\frac{dI(t)}{dt} = \frac{\pi \cos t\pi}{t} - \frac{\sin t\pi}{t^2}.$$

Then use this result, together with Leibniz's rule for the differentiation of an integral, to evaluate

$$\int_0^\pi x \sin\, tx\, dx.$$

35 Calculate $\dfrac{d}{dt}\displaystyle\int_{\sqrt{t}}^{t} \frac{\cos tx}{x} dx$ by using Leibniz rule.

2. Extreme Values

36 Test the following functions for maxima, minima and saddle points. Find the function values at these points.

a. $f(x, y) = x^2 + xy + y^2 + 3x - 3y + 4$.
b. $f(x, y) = x^2 + xy + 3x + 2y + 5$.
c. $f(x, y) = x^2 + xy + y^2 + 3y + 3$.
d. $f(x, y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$.
e. $f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy$.
f. $f(x, y) = 4xy - x^4 - y^4$.

37 Find all critical points of the function

$$f(x, y) = x^3 + y^3 - 3xy + 15,$$

and classify each one as a relative maximum, relative minimum, or saddle point.

38 Determine and classify the critical points (extrema) of the following function

$$g(x, y) = x^2 - Axy + y^2 + 7,$$

where A is a positive constant.
Discuss separately the cases $0 < A < 2$, $A > 2$ and $A = 2$.

39 Find the points on the ellipse $x^2 + 2y^2 = 1$ where $f(x, y) = xy$ has its extreme values.

40 Find the extreme values of $f(x, y) = x^2y$ on the line $x + y = 3$.

41 Use the method of Lagrange multipliers to find

a. the minimum value of $x + y$ subject to the constraints $xy = 16$, $x > 0$, $y > 0$.
b. the maximum value of xy subject to the constraint $x + y = 16$.

42 Find the dimensions of the closed circular can of smallest surface area whose volume is $16\pi\text{cm}^3$.

43 The temperature at the point (x, y) on a metal plate is $T(x, y) = 4x^2 - 4xy + y^2$. An ant on the plate walks around the circle of radius 5 centred at the origin. What are the highest and lowest temperatures encountered by the ant?

44 Find the point on the plane $x + 2y + 3z = 13$ closest to the point $(1, 1, 1)$.

45 Find points on the surface $z^2 = xy + 4$ closest to the origin.

46 The temperature at any point (x, y, z) in space is $T = 400xyz^2$. Find the highest temperature on the unit sphere $x^2 + y^2 + z^2 = 1$.

47 Find the maximum value of $f(x,y,z) = x^2 + 2y - z^2$ subject to the constraints $2x - y = 0$ and $y + z = 0$.

3. Vector Field Theory

48 Let $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{w} = 2\mathbf{j} + 3\mathbf{k}$. Find
a. $(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$; **b.** $\mathbf{u}(\mathbf{v}\cdot\mathbf{w})$; **c.** $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$;
d. $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$; **e.** $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$; **f.** $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$.

49 Find the equations of the straight lines that satisfy each of the following sets of conditions.
a. The line $\mathcal{L}$ passes through the points $P(3,3,-5)$ and $Q(2,-6,1)$;
b. The line $\mathcal{L}$ passes through the point $P(1,-1,1)$ and is perpendicular to the plane $2x + 3y - z = 4$.

50 Find the volume of the parallelepiped with one corner at P and sides PQ, PR and PS where P,Q,R,S are:
a. $P(0,1,-6), Q(-3,1,4), R(1,7,2), S(-3,0,4)$;
b. $P(1,1,1), Q(-2,1,6), R(3,5,7), S(0,1,6)$.

51 For any vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$, show that
a. $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$;
b. $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) + \mathbf{v}\times(\mathbf{w}\times\mathbf{u}) + \mathbf{w}\times(\mathbf{u}\times\mathbf{v}) = \mathbf{0}$.

52 For each of the following, determine $(\mathbf{F}\cdot\mathbf{G})'$ and $(\mathbf{F}\times\mathbf{G})'$:
a. $\mathbf{F} = (\cos 2t)\mathbf{i} + (\sin t)\mathbf{j} - e^{-t}\mathbf{k}$, $\mathbf{G} = 2t^2\mathbf{i} - 3t\mathbf{k}$;
b. $\mathbf{F} = 5t^2\mathbf{i} + t\mathbf{j} - t^3\mathbf{k}$, $\mathbf{G} = (\sin t)\mathbf{i} - (\cos t)\mathbf{j}$.

53 A particle moves along a curve whose parametric equations are

$$x(t) = e^{-t},\ y(t) = 2\cos 3t,\ z(t) = 2\sin 3t,$$

where t is the time.
a. Determine its velocity vector and acceleration vector.
b. Find the magnitudes of the velocity and acceleration at $t = 0$.

54 Compute $\nabla\psi$ and $\nabla\psi(P_0)$ for the given point P_0 for
a. $\psi = e^{xy} + z^2 x$, $P_0(0,0,4)$;
b. $\psi = x^2 y - \sin(zx)$, $P_0(1,-1,\pi/4)$.

55 Find the tangent plane and normal line to the surface S at the point P_0 :
a. $S : x^2 + y^2 + z^2 = 4$, $P_0(1,1,\sqrt{2})$;
b. $S : x^2 - 2y^2 + z^4 = 0$, $P_0(1,1,1)$.

56 The atmosphere pressure in a certain region of space is $P = xy^2 + yz^2 + zyx$. Find the rate of change of the pressure with respect to distance at the point $(1,1,4)$ in the region, in the direction of the vector $\mathbf{v} = \mathbf{j} - 3\mathbf{k}$.

57 Consider the scalar field

$$\phi(x,y,z) = -2xy + x\ln(y+z)$$

determine:
a. the direction and magnitude of the maximum rate of change of ϕ at $(1,3,-2)$;
b. the directional derivative of ϕ in the $(1,3,-2)$ direction at $(1,1,0)$.

58 Given a scalar field

$$\phi(x,y,z) = 2x^2 + 3y^2 + z^2$$

a. Calculate $\nabla\phi$;
b. Using **a.** calculate $\nabla\phi$ at the point $P(2,1,3)$;
c. Find a unit normal to $\phi(x,y,z) = 20$ at the point $P(2,1,3)$;
d. Calculate the directional derivative of ϕ at $P(2,1,3)$ parallel to the vector $\mathbf{a} = \mathbf{i} - 2\mathbf{k}$.

59 Compute $\nabla\cdot\mathbf{F}$ and $\nabla\times\mathbf{F}$ and verify that $\nabla\cdot(\nabla\times\mathbf{F}) = 0$ for
a. $\mathbf{F} = x^2 z\mathbf{i} - y\mathbf{j} + z^3\mathbf{k}$;
b. $\mathbf{F} = (\sinh x)\mathbf{i} + (\cosh y)\mathbf{j} - xyz\mathbf{k}$.

60 Compute $\nabla\psi$ and verify that $\nabla \times (\nabla\psi) = \mathbf{0}$ for
a. $\psi = \sin(xz) + \cos(yz)$;
b. $\psi = -4xy^3 + z^2x$.

61 Evaluate $\int_{(1,1)}^{(4,2)} (x+y)dx + (y-x)dy$ along:
a. the parabola $y^2 = x$;
b. the straight line segments from $(1,1)$ to $(1,2)$ and then to $(4,2)$.

62 Evaluate $\oint (2x - y + 4)dx + (5y + 3x - 6)dy$ around
a. a triangle in the xy-plane with vertices at $(0,0)$, $(3,0)$, $(3,2)$ traversed in the counterclockwise direction;
b. a circle of radius 4 with centre at $(0,0)$ traversed counterclockwise.

63 Calculate the work done by the force field $\mathbf{F}$ along the curve C if
a. $\mathbf{F} = 3xy\mathbf{i} - 2\mathbf{j}$ and C is the piece of the hyperbola $x^2 - y^2 = 1$, $z = 0$ from $(1,0,0)$ to $(2,\sqrt{3},0)$;
b. $\mathbf{F} = x^3\mathbf{i} - z\mathbf{j} + 2xy\mathbf{k}$ and C is given by

$$x = t^2,\ y = z = \sqrt{t}, \qquad 2 \le t \le 4.$$

4. Double Integrals

64 Evaluate the following double integrals:
a. $\displaystyle\int_0^2 \int_1^3 x^3y^2\,dy\,dx$; **b.** $\displaystyle\int_1^3 \int_2^3 (x^2 - 2xy + 2y^2)\,dy\,dx$.

65 Use double integration to find the area of the following regions:
a. the region bounded by $y = x^3$ and $y = x^2$;
b. the region bounded by $y = \sqrt{x}$, $y = x$ and $y = x/2$.

66 Integrate the following functions f over the given regions Ω:
a. $f(x,y) = xy$, Ω bounded by $y = 0$, $x = 2a$, and $x^2 = 4ay$;
b. $f(x,y) = x^2y + y^3$, $\Omega = \{(x,y)\,|\,x^2 + y^2 \le 1,\ x \ge 0,\ y \ge 0\}$.

67 Evaluate the following integrals by first changing the order of integration:
a. $\displaystyle\int_0^1 \int_{y^2}^1 2\sqrt{x}e^{x^2}\,dx\,dy$; **b.** $\displaystyle\int_0^1 \int_y^1 e^{x^2}dx\,dy$; **c.** $\displaystyle\int_0^1 \int_y^1 \sin(x^2)\,dx\,dy$;
d. $\displaystyle\int_{-1}^1 \int_{x^2}^{2-x^2} dy\,dx$; **e.** $\displaystyle\int_0^1 \int_x^{2x} x^2e^x\ dy\,dx$.

68 Evaluate using polar co-ordinates:
a. $\displaystyle\int_0^2 \int_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} x^2y^2\ dy\,dx$; **b.** $\displaystyle\int_0^2 \int_{-\sqrt{(2y-y^2)}}^{\sqrt{(2y-y^2)}} \sqrt{(x^2+y^2)}\ dx\,dy$.

69 **a.** Find the volume lying between the paraboloids $z = x^2 + y^2$ and $3z = 4 - x^2 - y^2$.
b. Find the volume lying inside both the sphere $x^2 + y^2 + z^2 = 2a^2$ and the cylinder $x^2 + y^2 = a^2$, with $a > 0$.

70 Given the integral

$$\int_0^4 \int_{3x}^{12} \sin(y^2)dy\,dx$$

a. make a sketch of the region of integration;
b. express the integral with the reverse order of integration;
c. hence evaluate it.

71 Given

$$\int_0^1 \int_{\sqrt{x}}^1 \sqrt{1+y^3}dy\,dx$$

a. make a sketch of the region of integration;
b. express the integral with the reverse order of integration;
c. hence evaluate it (leaving your answer in surd form).

72 Find the centroid of the region in the first quadrant bounded by the x-axis, the parabola $y^2 = 2x$, and the line $x + y = 4$.

73 Find the centroid of the region cut from the first quadrant by the circle $x^2 + y^2 = a^2$.

74 Find the centre of mass of a thin triangular plate bounded by the y-axis and the lines $y = x$ and $y = 2 - x$ if $\delta(x, y) = 6x + 3y + 3$.

75 Find the centre of mass and the moment of inertia about the x-axis of a thin plate bounded by the curves $x = y^2$ and $x = 2y - y^2$ if the density at the point (x, y) is given by $\delta(x, y) = y + 1$.

76 Find the centre of mass of a thin plate bounded by the semi-circle $y = \sqrt{a^2 - x^2}$, the lines $x = \pm a$ and the line $y = -a$ if the density $\delta(x, y)$ is given by

a. k (some constant), **b.** $y + a$, **c.** $x + a$.

77 Find the volume of the tetrahedron bounded by the co-ordinate planes and the plane $z = 2 - 2x - y$.

78 Find the volume inside the cylinder $x^2 + y^2 = 16$, cut off above by the plane $z = 5$ and below by the surface $z = \dfrac{(x^2 + y^2)}{8}$.

79 The solid S is bounded above by the sphere $z = \sqrt{2a^2 - x^2 - y^2}$ and below by the cone $z = \sqrt{x^2 + y^2}$. Sketch the solid and find its volume.

5. Ordinary differential equations.

80 Find the general solution to the following 1st order differential equations.

a. $2e^x + \dfrac{dy}{dx}(1 - e^x)\tan y = 0$;

b. $\sec^2 x \tan y + \dfrac{dy}{dx}\sec^2 y \tan x = 0$;

c. $(x^2 + 1)\dfrac{dy}{dx} + 2xy - 4x^2 = 0$;

d. $x^3 - 2y + 3x^2 y - x^3\dfrac{dy}{dx} = 0$;

81 Use the substitution $v = \dfrac{y}{x}$ to:

a. find the general solution to $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$.

b. solve the initial value problem

$$yy' = x^3 + \frac{y^2}{x}; \quad y(2) = 6.$$

82 Use the substitution $v = y + x$ to find the general solution of $\dfrac{dy}{dx} = (y + x)^2$.

83 Solve the following differential equations;

a. $y' = \dfrac{xy + 2}{1 - x^2}$, $y(0) = 1$;

b. $yy' = x^2 + \operatorname{sech}^2 x$, $y(0) = 4$.

84 Give the general solution to the following 2nd order differential equations.

a. $2y''(x) + y'(x) - 6y = 0$;

b. $y''(x) + 4y'(x) + 53y = 0$;

c. $y''(x) - 4y'(x) + 4y = 0$.

85 Solve

a. $y'' + 3y' + 2y = 30e^{4x}$;

b. $y'' - 4y' + 4y = xe^{3x}$;

c. $y''(x) - 4y'(x) + 3y = 9x^2 + 2e^{3x}$.

86 A forced vibrating system is represented by

$$y'' + 3y' + 2y = 5\cos t$$

where $r(t) = 5\cos t$ is the driving force and $y(t)$ is the displacement from the equilibrium position. Find the motion of the system corresponding to the following initial displacement and velocity

$$y(0) = \frac{1}{2}, \quad y'(0) = 1.$$

Then find the steady-state oscillations (that is, the response of the system after a sufficiently long time).

87 Solve

$$\frac{d^2x}{dt^2} + 9x = 12\sin 3t, \; x(0) = 5, \; \frac{dx}{dt}(0) = 4.$$

6. Matrices

88 Let

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}.$$

Find AB and BA.

89 Find the characteristic equation, the eigenvalues and the associated eigenvectors for the matrices

a. $A = \begin{pmatrix} 0 & -1 & -3 \\ 2 & 3 & 3 \\ -2 & 1 & 1 \end{pmatrix}$ **b.** $B = \begin{pmatrix} -6 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix}$.

90 A curve has equation $x^2 + 8xy + 7y^2 = 36$. Find an orthogonal matrix P so that

$$\begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} X \\ Y \end{pmatrix}$$

will refer the equation to the principal axes of the curve and hence write down the equation in terms of X and Y. Give the x, y co-ordinates of the points on the curve closest to the origin.

91 **a.** Show that the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

are 1, -4 and 3 and then find the associated eigenvectors.
b. Hence
i) express the equation of the surface

$$x^2 - 2y^2 + z^2 + 6xy - 2yz = 16$$

in terms of its principal axes X, Y and Z and
ii) write out an orthogonal matrix P such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}.$$

c. Deduce from your results the shortest distance from the origin to the surface described in (b).

92 Use eigenvalue methods to find the general solution to the system of differential equations

$$\begin{aligned}\frac{dx}{dt} &= 7x + y + z\\ \frac{dy}{dt} &= 3x + y + 2z\\ \frac{dz}{dt} &= x + 3y + 2z\end{aligned}$$

93 **a.** By solving for the zeros of the characteristic polynomial show that the eigenvalues of the matrix

$$A = \begin{pmatrix} 3 & 2 & 2\\ 2 & 2 & 0\\ 2 & 0 & 4\end{pmatrix}$$

are 0, 3 and 6.

b. Find the eigenvectors corresponding to the eigenvalues found in **a.**

c. Find an orthogonal matrix P such that

$$D = P^{-1}AP$$

is a diagonal matrix.

d. Using the results of a), b), c) find the solution to the system of differential equations

$$\begin{aligned}\frac{dx_1}{dt} &= 3x_1 + 2x_2 + 2x_3\\ \frac{dx_2}{dt} &= 2x_1 + 2x_2\\ \frac{dx_3}{dt} &= 2x_1 + 4x_3\end{aligned}$$

subject to the initial conditions $x_1(0) = 3$, $x_2(0) = 1$, $x_3(0) = 4$.

7. The Laplace Transform

94 Find, by direct integration, the Laplace transforms of

a. $5t + 3$ **b.** $\cos(wt)$.

95 Use tables to find the Laplace transforms of

a. $t^2 + 2t + 3$

b. $\sin 5t$

c. e^{3t-4}

d. te^{2t}

e. t^6e^{4t}

f. $e^t \sin t$

g. $t \cos wt$

h. $4t^2e^t$.

96 **a.** Sketch the function $g(t) = 2t - 2u(t-1)$

b. Find $\mathcal{L}(g(t))$.

97 Find the Laplace transform of

a. $(t-5)^3u(t-5)$

b. $\cos 3(t-4)u(t-4)$

98 Use tables to find

a. $\mathcal{L}^{-1}\left(\dfrac{1}{25s^2}+\dfrac{s}{s^2+25}\right)$

b. $\mathcal{L}^{-1}\left(\dfrac{1}{s^2+(\pi/2)^2}\right)$

c. $\mathcal{L}^{-1}\left(\dfrac{s-2}{s^2-4s+5}\right)$

d. $\mathcal{L}^{-1}\left(\dfrac{\pi}{(s+\pi)^2}\right)$

e. $\mathcal{L}^{-1}\left(\dfrac{1}{(s+3)(s+2)}\right)$

f. $\mathcal{L}^{-1}\left(\dfrac{12s}{s^2+5s+4}\right)$

g. $\mathcal{L}^{-1}\left(\dfrac{e^{-3s}}{s^2}\right)$

h. $\mathcal{L}^{-1}\left(\dfrac{e^{-2s}s}{s^2+9}\right)$

i. $\mathcal{L}^{-1}\left(\dfrac{3e^{-s}}{(s-2)^2}\right)$

99 Using Laplace transforms solve

$$y''+4y=0, \qquad y(0)=2, \qquad y'(0)=-8.$$

100 Use the method of Laplace transforms to solve the differential equation

$$\frac{d^2x}{dt^2}-5\frac{dx}{dt}+6x=24e^{-t}$$

when $x=3$ and $\dfrac{dx}{dt}=2$ at $t=0$.

101 Solve $y''-4y'+5y=0, \quad y(0)=1, \ y'(0)=2$

102 Use the Laplace transform method to find a solution to the system

$$\frac{dx}{dt}+2y-x=0$$
$$\frac{dy}{dt}-2x-y=0$$

satisfying the initial conditions $x=1$ and $y=0$ when $t=0$.

103 Use the Laplace transform method to solve the initial value problem

$$y''-3y'+2y=r(t), \qquad y(0)=1, \ y'(0)=3,$$

where $r(t)$ is as shown below.

104 A particle of mass m moves along the x axis. At time $t = 0$ it is at the origin and moving with velocity V, when a constant force amplitude F is applied for a time of a, after which it is removed.

Find the position of the particle at any time, using the Laplace transform method. That is, solve the following problem:

$$m\frac{d^2x}{dt^2} = (1 - u(t-a))F \qquad \text{for} \quad t > 0,$$

where u is the Heaviside step function, and

$$x(0) = 0 \qquad \text{and} \qquad \frac{dx}{dt}(0) = V.$$

8. Fourier Series

105 State which of the following functions are odd, even, both or neither.
a. $|x|$
b. $x\cos(nx)$
c. $\sin(x) + \cos(x)$
d. c, where c is a constant
e. $\ln(1+e^x) - x/2$
f. $\sin^2(x)$.

106 The following functions f are assumed to be periodic with period 2π. Sketch for $-4\pi \le x \le 4\pi$. Are they odd, even or neither?
a. $f(x) = x|x|;\ -\pi < x < \pi$
b. $f(x) = e^{|x|} : -\pi < x < \pi$
c. $f(x) = \begin{cases} x; & -\pi/2 < x < \pi/2 \\ 0; & \pi/2 < x < 3\pi/2 \end{cases}$

107 Find the Fourier series for

$$f(x) = \begin{cases} 5 & -\pi < x \le 0 \\ 3 & 0 < x \le \pi. \end{cases}$$

108 For the function g given by

$$g(x) = \begin{cases} 1 & 0 < x < 1 \\ 4 - 2x & 1 \le x \le 2. \end{cases}$$

i) Sketch over $(-10, 10)$ the function represented by the half-range Fourier sine series.
ii) Make a separate sketch over $(-10, 10)$ of the function represented by the half-range Fourier cosine series.

109 Find the Fourier series of $L(x) = 3(x+1)$ for $-2 < x < 2$. Find a result on an infinite series by considering your answer at $x = 5$.

110 Periodically extend the function

$$f(t) = e^{-t}, \qquad 0 < t < 1,$$

in an odd manner over $(-1 < t < 0)$ and find its Fourier series. Plot $f(t)$ for $(-2 < t < 2)$ and state the value of the Fourier series representation at $t = 0$.

111 Let

$$f(x) = e^{-x} \qquad \text{for} \quad 0 \le x \le 1,$$

and suppose that f is extended to an even function with period 2; thus

$$f(-x) = f(x), \qquad f(x+2) = f(x)$$

for all x.

a. Sketch the graph of $f(x)$ for $-2 \le x \le 2$
b. Find the coefficients in the cosine half-range expansion

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x).$$

You may assume that

$$\int e^{-x}\cos(bx)dx = \frac{e^{-x}[b\sin(bx)-\cos(bx)]}{1+b^2} + \text{constant}.$$

c. Compute the numerical values of a_0 and a_1, and use these to sketch the graph of

$$S_1(x) = a_0 + a_1\cos(\pi x)$$

for $-2 \le x \le 2$. (If you don't have a calculator, then use the approximations $e^{-1} \approx 0.4$ and $\pi^2 \approx 10$.) Are your graphs of $f(x)$ and $S_1(x)$ roughly similar in shape?

112 A vibrating system is governed by the differential equation

$$\frac{d^2x}{dt^2} + 50x = F(t), \tag{1}$$

where t is the time, $x(t)$ is the displacement from equilibrium and $F(t)$ is the applied force function.
a. When the function $F(t)$ is represented by the series

$$F(t) = b_0 + \sum_{n=1}^{\infty} b_n \sin nt \tag{2}$$

find a series which is a particular integral of the differential equation (1) given above.
b. For the case of the following periodic force

$$F(t) = \begin{cases} 2 & \text{for } 0 < t < \pi \\ 0 & \text{for } -\pi < t < 0 \end{cases}$$

with $F(t+2\pi) = F(t)$ for all t, write down an integral formula for the b_n in the Fourier series (2) and evaluate the integral.
Hence find an infinite series which is a particular integral of the differential equation (1).
c. By tabulating the amplitudes of the various components of the input forcing function and the output displacement of the system, or otherwise, determine which component of the forcing function gives the largest contribution to the observed output displacement. What is this phenomenon called?

9. Partial Differential Equations

113 a. Use D'Alembert's method to find a solution in terms of arbitrary functions for

$$\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial y^2} = 0.$$

b. Determine the particular solution satisfying

$$u(x,0) = 0 \qquad \text{and} \qquad u_y(x,0) = 8\sin 2x.$$

114 a. By finding the values of λ for which $u = F(x - \lambda y)$ is a solution of the p.d.e., find a solution in terms of two arbitrary functions for

$$7\frac{\partial^2 u}{\partial x^2} - 8\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0.$$

b. Determine the particular solution satisfying

$$u(x,0) = 0 \qquad \text{and} \qquad u_y(x,0) = 9e^{-x}.$$

115 **a.** The temperature $u(x,t)$ in a bar of unit length obeys the equation

$$\frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2}.$$

The bar has its ends maintained at zero temperature, that is,

$$u(0,t) = 0 \qquad \text{for all} \quad t \geq 0$$

and

$$u(1,t) = 0 \qquad \text{for all} \quad t \geq 0.$$

The initial temperature distribution is

$$u(x,0) = \sin 2\pi x - \frac{1}{3}\sin 4\pi x \quad \text{for} \quad 0 \leq x \leq 1.$$

Obtain the solution $u(x,t)$ by using the method of separation of variables.

116 Solve

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \quad \text{for } 0 < x < L,\ t > 0,$$

subject to

i. $u = 0$ when $x = 0 \quad \forall t$

ii. $u = 0$ when $x = L \quad \forall t$

iii. $u = k \sin \dfrac{4\pi x}{L}$ when $t = 0$.

iv. $\dfrac{\partial u}{\partial t} = 0$ when $t = 0$.

117 The temperature in a bar of unit length obeys the heat equation

$$\frac{\partial v}{\partial t} = \frac{1}{4}\frac{\partial^2 v}{\partial x^2},$$

where $v(x,t)$ is the temperature. The bar has an initial temperature distribution

$$v(x,0) = \begin{cases} \alpha & 0 \leq x < 1/2 \\ 0 & 1/2 \leq x \leq 1 \end{cases}$$

and is insulated so that the flux of heat at each end is zero;

$$\frac{\partial v}{\partial x} = 0 \qquad \text{at} \qquad x = 0 \quad \text{and} \quad x = 1$$

for all t. Using the method of separation of variables, obtain the solution $v(x,t)$. Plot the temperature distribution at $t = 0$ and as $t \to \infty$, and explain why v is non-zero in the $t \to \infty$ case.

118 The steady-state distribution of heat in a slab of height h is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for} \quad x > 0,\ 0 < y < h,$$

with boundary conditions

i. $u = 0$ when $y = 0$, all $x > 0$;

ii. $u = 0$ when $y = h$, all $x > 0$;

iii. $u \to 0$ when $x \to \infty$, for $0 < y < h$;

iv. $u = 1$ when $x = 0$, for $0 < y < h$.

Use the method of separation of variables to find a solution for $u(x,y)$.

SOME ANSWERS

2 **a.** $f_{xx} = -f_{yy} = \frac{-2(x^2-y^2)}{(x^2+y^2)^2}$; $f_{xy} = \frac{-4xy}{(x^2+y^2)^2}$. **b.** $f_{xx} = 2y - y\sin x, f_{yy} = -\cos y, f_{xy} = 2x + \cos x$.
c. $\frac{2xy}{(x^2+y^2)^2}, \frac{-2xy}{(x^2+y^2)^2}, \frac{y^2-x^2}{(x^2+y^2)^2}$.

5 **a.** $1/(1+\pi)$. **b.** 0. **6** $4x\cos 2t - 4yte^{y^2}\sin t^2$. **7** $-2xy - 2y^2 + 2x^2$.

8 2. **9** 99. **10** 0. **15** **a.** $x - \frac{x^3}{6}$, **b.** $1 - (x-\pi/2)^2/2$.

18 **a.** $1 - 2x^2 + 2xy - \frac{y^2}{2} + \cdots$, **b.** $-2x + (y+\pi/2) + \cdots$.

19 $e\{1 + (x-1) + y + \frac{1}{2}(x-1)^2 + (x-1)y + \cdots\}$.

20 $\frac{1}{\sqrt{2}} - \sqrt{2}\left(x - \frac{\pi}{4}\right) + \frac{1}{\sqrt{2}}\left(y - \frac{\pi}{4}\right) - \sqrt{2}\left(x - \frac{\pi}{4}\right)^2 + \sqrt{2}\left(x - \frac{\pi}{4}\right)\left(y - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(y - \frac{\pi}{4}\right)^2$

21 $2(x-1) - (x-1)^2 + y^2, 0.2$. **22** $e^{-2}\left[1 - (x-2) + \frac{(x-2)^2}{2} - \frac{(y-\frac{\pi}{2})^2}{2}\right]$, 0.14659.

23 2.95. **24** $|dT| \le \frac{8}{100}$. **25** $|100dv/v| \le 2.42\%$. **26** 0.1%.

27 0.024. **28** 16%. **29** **a.** 3.6, **c.** .0408.

31 **a.** $\frac{2ka}{(a^2+k^2)^2}$. **b.** $\frac{a^2-k^2}{(a^2+k^2)^2}$. **33** $\frac{\pi\alpha}{(\alpha^2-1)^{3/2}}$. **34** $\frac{\sin t\pi}{t^2} - \frac{\pi\cos tx}{t}$.

35 $\frac{2\cos t^2}{t} - \frac{3\cos t^{\frac{3}{2}}}{2t}$.

36 **a.** Minimum of -5 at $(-3,3)$. **b.** Saddle point at $(-2,1)$. **c.** Minimum of 0 at $(-1,-2)$. **d.** Minimum of -6 at $(2,-1)$. **e.** Saddle point at $(1,-1)$; minimum of 0 at $(0,0)$. **f.** Saddle point at $(0,0)$; maximum of 2 at $(1,1)$ and $(-1,-1)$.

37 Saddle point at $(0,0)$. Minimum of 14 at $(1,1)$.

38 $0 < A < 2$ local minimum at $(0,0,7)$; $A > 2$ saddle point at $(0,0,7)$; $A = 2$ local minimum on line $x = y$.

39 Maximum at $(\pm 1/\sqrt{2}, \pm 1/2)$, minimum at $(-1/\sqrt{2}, 1/2)$ and $(1/\sqrt{2}, -1/2)$.

40 Local minimum of 0 at $(0,3)$, local maximum of 4 at $(2,1)$.

41 **a.** Minimum of 8 at $(4,4)$ **b.** maximum of 64 at $(8,8)$. **42** $r = 2$cm, $h = 4$cm.

43 $T = 0$ at $(\sqrt{5}, 2\sqrt{5})$ or $(-\sqrt{5}, -2\sqrt{5})$, $T = 125$ at $(2\sqrt{5}, -\sqrt{5})$ or $(-2\sqrt{5}, \sqrt{5})$.

44 $(3/2, 2, 5/2)$. **45** $(0,0,\pm 2)$ are each closest to the origin. **46** 50.

47 $x = 2/3$, $y = 4/3$, $z = -4/3$.

48 **a.** $-10\mathbf{j} - 15\mathbf{k}$, **b.** $6\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$, **c.** 30. **d.** 30. **e.** $-7\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, **f.** $-\mathbf{i} + 7\mathbf{j} + 16\mathbf{k}$.

49 **a.** $x = 3 - t, y = 3 - 9t, z = -5 + 6t$. **b.** $\mathbf{r} = (1+2t)\mathbf{i} + (-1+3t)\mathbf{j} + (1-t)\mathbf{k}$.

50 **a.** 34. **b.** 40.

52 **a.** $(-3\sin t - 3t\cos t)\mathbf{i} + (3\cos 2t - 6t\sin 2t - 4te^{-t} + 2t^2e^{-t})\mathbf{j} + (-4t\sin t - 2t^2\cos t)\mathbf{k}$.
b. $(-3t^2\cos t + t^3\sin t)\mathbf{i} - (3t^2\sin t + t^3\cos t)\mathbf{j} + (5t^2\sin t - \sin t - 11t\cos t)\mathbf{k}$.

53 **a.** $\mathbf{v} = -e^{-t}\mathbf{i} - 6\sin 3t\mathbf{j} + 6\cos 3t\mathbf{k}$. $\mathbf{a} = e^{-t}\mathbf{i} - 18\cos 3t\mathbf{j} - 18\sin 3t\mathbf{k}$. **b.** $\sqrt{37}, \sqrt{325}$.

54 **a.** $16\mathbf{i}$ **b.** $(-2 - \frac{\pi\sqrt{2}}{8})\mathbf{i} + \mathbf{j} - \frac{\sqrt{2}}{2}\mathbf{k}$.

55 **a.** $x = 1 + 2t,\ y = 1 + 2t,\ z = \sqrt{2} + 2\sqrt{2}t$. **b.** $x = 1 + 2t,\ y = 1 - 4t,\ z = 1 + 4t$.

56 $-\frac{5}{\sqrt{10}}$. **57** **a.** $\sqrt{38}$. **b.** $-\frac{7}{\sqrt{14}}$. **61** **a.** $11\frac{1}{3}$. **b.** 14.

62 **a.** 12. **b.** 64π. **63** **a.** $\sqrt{3}$. **b.** $\frac{49013}{3}$. **64** **a.** 104/3. **b.** 14.

65 **a.** $\frac{1}{12}$ **b.** 7/6. **66** **a.** $a^4/3$ **b.** 1/5.

67 **a.** $(e-1)$ **b.** $(e-1)/2$ **c.** $(1-\cos 1)/2$ **d.** 8/3 **e.** $e^2/4 - e + 7/4$.

68 **a.** $4\pi/3$ **b.** 32/9. **69** **a.** $\frac{2\pi}{3}$. **b.** $4\pi a^3\left[2\sqrt{2} - 1\right]/3$. **70** $\frac{1}{6}(1 - \cos 144)$.

71 $\frac{2}{9}(2\sqrt{2} - 1)$ **72** $\left(\frac{64}{35}, \frac{5}{7}\right)$ **73** $\left(\frac{4a}{3\pi}, \frac{4a}{3\pi}\right)$ **74** $\left(\frac{3}{8}, \frac{17}{16}\right)$.

75 $\left(\frac{8}{15}, \frac{8}{15}\right)$, $I_x = \frac{1}{6}$. **78** 64π.

80 **a.** $\sec y = C(e^x - 1)^2$ **b.** $\tan x \tan y = C$. **c.** $y = \dfrac{\frac{4}{3}x^3 + c}{x^2+1}$. **d.** $y = \dfrac{x^3}{2} + Cx^3e^{1/x^2}$.

83 **a.** $y\sqrt{1-x^2} = 2\sin^{-1}x + 1$. **b.** $3y^2 = 2x^3 + 6\tanh x + 48$.

84 **a.** $y = Ae^{\frac{3x}{2}} + Be^{-2x}$. **b.** $y = e^{-2x}(A\cos 7x + B\sin 7x)$. **c.** $y = (C_1 + C_2x)e^{2x}$.

85 **a.** $y = Ae^{-x} + Be^{-2x} + e^{4x}$. **b.** $y = (A+Bx)e^{2x} + (x-2)e^{3x}$. $y = Ae^x + (x+B)e^{3x} + 3x^2 + 8x + \frac{26}{3}$.

86 $y = \frac{1}{2}(\cos t + 3\sin t)$. **87** $x = (5-2t)\cos 3t + 2\sin 3t$.

88 $AB = 0 \qquad BA = \begin{bmatrix} -11 & 6 & 1 \\ -22 & 12 & -2 \\ -11 & 6 & -1 \end{bmatrix}$.

89 **a.** $4, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix};\ 2, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix};\ -2, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ **b.** $-7, \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix};\ -1, \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix};\ 3, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

90 $\begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}, \quad \frac{X^2}{4} - \frac{Y^2}{36} = 1, \ \pm\left(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right)$.

91 **b.** i) $X^2 + 3Y^2 - 4Z^2 = 16$. ii) $P = \begin{pmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{14}} & -\frac{3}{\sqrt{35}} \\ 0 & -\frac{2}{\sqrt{14}} & \frac{5}{\sqrt{35}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{35}} \end{pmatrix}$ **c.** $\frac{4}{\sqrt{3}}$

92 $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = a \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{-t} + b \begin{pmatrix} 4 \\ -5 \\ -11 \end{pmatrix} e^{3t} + c \begin{pmatrix} 9 \\ 5 \\ 4 \end{pmatrix} e^{8t}$

93 **b.** $0, \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}, 3, \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, 6, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ **c.** $\frac{1}{3}\begin{pmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{pmatrix}$ **d.** $\underline{X} = \frac{1}{3}\left[\begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix} e^{3t} + \begin{pmatrix} 10 \\ 5 \\ 10 \end{pmatrix} e^{6t}\right]$

94 **a.** $\frac{5}{s^2} + \frac{3}{s}$, **b.** $\frac{s}{w^2 + s^2}$.

95 **a.** $\frac{2}{s^3} + \frac{2}{s^2} + \frac{3}{s}$. **b.** $\frac{5}{w^2 + 25}$ **c.** $\frac{e^{-4}}{s-3}$ **d.** $\frac{1}{(s-2)^2}$ **e.** $\frac{6!}{(s-4)^7}$ **f.** $\frac{1}{1+(s-1)^2}$ **g.** $\frac{s^2-w^2}{(s^2+w^2)^2}$ **h.** $\frac{8}{(s-1)^3}$

96 $\frac{2}{s^2} - \frac{e^{-s}}{s}$ **97** **a.** $\frac{3!e^{-5s}}{s^4}$ **b.** $\frac{se^{-4s}}{s^2+9}$

98 **a.** $\frac{t}{25} + \cos 5t$ **b.** $\frac{2}{\pi}\sin\frac{\pi t}{2}$ **c.** $e^{2t}\cos t$ **d.** $\pi e^{-\pi t} t$ **e.** $-e^{-3t} + e^{-2t}$ **f.** $16e^{-4t} - 4e^{-t}$ **g.** $(t-3)u(t-3)$ **h.** $\cos 3(t-2)u(t-2)$ **i.** $3e^{2(t-1)}(t-1)u(t-1)$

99 $y = 2\cos 2t - 4\sin 2t$ **100** $x = 2e^{-t} + 2e^{3t} - e^{2t}$ **101** $y = e^{2t}\cos t$

102 $x = e^t\cos 2t; \qquad y = e^t \sin 2t$.

103 $y = u(t-1)[1/2 + e^{2(t-1)}/2 - e^{t-1}] - u(t-2)[1/2 + e^{2(t-2)}/2 - e^{t-2}] + [2e^{2t} - e^t]$.

104 $x = \frac{Ft^2}{2m} + Vt, \quad t < a; \qquad x = \frac{Ft^2}{2m} - \frac{F}{2m}(t-a)^2 + Vt, \qquad t > a.$

107 $4 + \sum_{n=1}^{\infty} \frac{2}{n\pi}(\cos n\pi - 1)\sin nx$. **109** $3 + \sum_{n=1}^{\infty} \frac{-12}{n\pi}\cos n\pi \sin\frac{n\pi x}{2}, \ \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\cdots$.

110 $e^{-t} = \sum_{n=1}^{\infty} \frac{2n\pi}{e}\left[\frac{e-(-1)^n}{1+n^2\pi^2}\right]\sin n\pi t$ **111** $a_0 = 1 - \frac{1}{e}, \ a_n = \frac{2}{1+n^2\pi^2}\left[1 - \frac{(-1)^n}{e}\right]$

112 **a.** $\frac{b_0}{50} + \sum_{n=1}^{\infty}\frac{b_n}{50-n^2}\sin nt$ **b.** $\frac{1}{50} + \sum_{k=1}^{\infty}\frac{4}{(2k-1)\pi(50-(2k-1)^2)}\sin(2k-1)t$ **c.** $k = 4$

113 $u = -4\cos(2x+y) + 4\cos(2x-y)$ **114** $u = -\frac{3}{2}e^{-(x+7y)} + \frac{3}{2}e^{-(x+y)}$

115 $u = e^{-\pi^2 t}\sin 2\pi x - \frac{1}{3}e^{-4\pi^2 t}\sin 4\pi x$. **116** **iii.** $u = k\sin\frac{4\pi x}{L}\cos\frac{4\pi ct}{L}$

117 $v = \frac{\alpha}{2} + \frac{2\alpha}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin\frac{(2k+1)\pi}{2}\, e^{-(2k+1)^2\pi^2 t/4}\cos(2k+1)\pi x.$

118 $u = \sum_{k=0}^{\infty}\frac{4}{\pi}\left(\frac{1}{2k+1}\sin\frac{(2k+1)y}{h}\, e^{-(2k+1)\pi x/h}\right)$
