

MATH 1231 MATHEMATICS 1B CALCULUS.

Section 6: - Slicing Techniques and Further Applications

The objective of this section is to become familiar with the theory and application of slicing techniques and their application to

- volume;
- area;
- surface area;
- and arc-length.

By the end of this section students will be able to solve a range of problems involving the above ideas.

1. Volumes by Slicing:

Recall, that the volume of any prism (i.e. a solid with uniform cross section) is given by the area of the cross-section \times the height. We can exploit this idea for solids which do not have uniform cross-section.

Suppose we put an axis through the solid (in whatever place is most convenient) and slice the solid perpendicular to this axis at a point x where $a \leq x \leq b$. Let $V(x)$ be the volume of the solid from a to x , and $A(x)$ be the cross-sectional area at the point x . Thus the volume of a slice from x to $x + \Delta x$ will be $V(x + \Delta x) - V(x)$ which will be approximately $A(x)\Delta x$. Thus

$$A(x) \approx \frac{V(x + \Delta x) - V(x)}{\Delta x}.$$

Now if the limit as $\Delta x \rightarrow 0$ exists, we have

$$A(x) = \frac{dV}{dx}$$

and so, if $A(x)$ is integrable, we have

$$\text{Volume of the solid} = \int_a^b A(x) dx.$$

From this we can obtain the standard formula for volume of revolution.

Theorem: If f^2 is integrable, the volume of the solid obtained by rotating the curve

$$y = f(x)$$

(for $f(x) \geq 0$) about the x -axis between $x = a$ and $x = b$ is given by

$$\int_a^b \pi(f(x))^2 dx.$$

Proof: The cross section of a slice at distance x from the origin is given by $\pi(f(x))^2$ and result follows.

Ex. Find the volume of the solid obtained by rotating the curve $y = x^2$ about the x -axis between the points $x = 0$ and $x = 2$.

Similar Areas: We can use the idea of similar shapes to find volumes.

Ex. Find the volume of a right circular cone with height h and radius r .

The practical steps involved are:

1. Draw a clear sketch of the solid and a typical cross-section.
2. Find $A(x)$.
3. Find a and b .
4. Integrate.

Ex. Find the volume of the solid generated when the area bounded by the x -axis and the curve $y = -x^2 + 4x - 3$ is rotated about the y -axis.

Ex. Find the volume of the torus obtained by rotating $(x - a)^2 + y^2 = b^2$ (where $0 < b < a$) around the y -axis.

2. Arc Length:

We seek to find a formula for arc length of a given curve, $y = f(x)$.

Take a small piece of arc length Δs between

the points x and $x + \Delta x$ on the curve and draw an approximate triangle there. By Pythagoras' theorem we have

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2. \quad (*)$$

Now dividing by $(\Delta x)^2$ and assuming the necessary limits exist, we have, using the same basic idea we used for slicing,

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

provided f is differentiable and the integral exists.

Ex. Find the arc length of the curve

$$y = \frac{1}{4}x^2 - \frac{1}{2}\log x$$

from $x = 1$ to $x = 2$.

Parametric Formula for Arc Length.

Ex. Find the arc length of the cycloid

$$x(t) = a(t - \sin t), \quad y(t) = a(1 - \cos t)$$

for $0 \leq t \leq 2\pi$.

Ex. Find the arc length of the curve

$$x = \int_0^t \frac{\sin s}{\sqrt{s}} ds, \quad y = \int_0^t \frac{\cos s}{\sqrt{s}} ds,$$

for $0 \leq t < 10$.

Surface Area.

The problem of finding the surface area of a solid is not easy. The general question will be left 'till second year. We will instead concentrate on the problem of finding the surface area of a solid of revolution (about the x -axis).

The formula we will obtain is essentially based on the following simple result from school, that the area of the *frustum* of a cone is $\pi(R + r)s$ where R and r are the radii and s is a the slant height.

Suppose that $y = f(x)$ is rotated around the x -axis between $x = a$ and $x = b$. We slice the surface into strips and approximate each strip by a frustum of a cone having radii y and $y + \Delta y$, with slant height being a 'small' piece of arc length which we can approximate as $\sqrt{(\Delta x)^2 + (\Delta y)^2}$.

Thus the area is

$$\pi(2y + \Delta y)\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x.$$

Summing the areas of the strip and letting $\Delta x, \Delta y \rightarrow 0$, we obtain

$$\text{Surface Area} = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Ex. Find the the surface area of the cone radius r height h . This is done by rotating the line $y = \frac{rx}{h}$, where $0 \leq x \leq h$, about the x - axis. We get $\pi r \sqrt{r^2 + h^2} = \pi r l$, where l is the slant height of the cone.

Ex. Find the surface area of the solid obtained by rotating $y = \sqrt{2-x}$ between $x = 0$ and $x = 2$ about the x -axis.

Ex. Find the surface area of the solid obtained by rotating $y = \frac{x^2}{2}$ about the x -axis between $x = 0$ and $x = 1$.

Average Value of a Function;

Suppose that f is an integrable function defined on $[a, b]$. Define

$$F(x) = \int_a^x f(t) dt.$$

From MATH1131, we know that F is differentiable and so we can apply the Mean Value Theorem to F on $[a, b]$ to obtain

$$\frac{F(b) - F(a)}{b - a} = F'(c), \quad \text{for some } c \in (a, b).$$

Thus

$$\frac{1}{b - a} \int_a^b f(t) dt = f(c)$$

or

$$\int_a^b f(t) dt = (b - a)f(c).$$

In other words, c is the point in (a, b) where the area under the curve is simply equal to the base length times the function height at c .

We call $f(c) = \frac{1}{b-a} \int_a^b f(t) dt$ the *average value* \bar{f} of the integrable function f . That is,

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Ex. The temperature during the afternoon t hours after noon is given by

$$25 + 2t - \frac{t^2}{3} \text{ } ^\circ\text{C}.$$

Find the average temperature between noon and 5p.m.

Centres of Mass:

Consider two objects located at x_1 and x_2 of mass m_1 and m_2 respectively on a stationary beam of negligible mass, as shown. Where do we put the fulcrum so that the system balances?

Let \bar{x} be the point, then

$$(\bar{x} - x_1)m_1 = (x_2 - \bar{x})m_2.$$

Solving, we have

$$\bar{x} = \frac{x_1m_1 + x_2m_2}{m_1 + m_2}.$$

The quantity in the numerator is called *the first moment*, and the bottom is simply the total mass of the system. The point \bar{x} is called the *centre of mass* of the system. In other words, the

$$\text{centre of mass} = \frac{\text{first moment}}{\text{total mass}}$$

.

The idea generalises in the obvious way. If we have n particles with position vectors

$$\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$$

with masses

$$m_1, m_2, \dots, m_n,$$

then the centre of mass is given by

$$\bar{\mathbf{x}} = \frac{m_1\mathbf{x}_1 + \dots + m_n\mathbf{x}_n}{m_1 + \dots + m_n}.$$

This means that in many instances the system of particles may be replaced by a single particle of mass $m_1 + \dots + m_n$ at the point $\bar{\mathbf{x}}$.

Ex. In \mathbf{R} , a massless beam carries a mass of 3kg, 2kg and 5kg at distances 1,2 and 3m respectively from the left hand end.
Calculate the centre of mass.

We can also consider the problem of finding the centre of mass in a continuous medium, such as a thin rod, an area or a solid. In this case, we have a continuous body (rather than a set of points) and some function (possibly constant) which gives the density of the body at any given point. We will only consider the case in \mathbf{R} . For higher dimensions, the problems is best solved using double and triple integrals which are covered in second year.

Consider a thin rod of length L and density $\rho(x)$.

The density is the mass per unit length, so if we take a small slice of length Δx at a point x , the mass will be given by $\Delta M \approx \rho(x)\Delta x$ (mass/unit length \times length). Summing and taking limits we have

$$\text{total mass} = \int_0^L \rho(x) dx.$$

The x -moment of the element Δx at x with mass $\rho(x)\Delta x$ is given by $x\rho(x)\Delta x$, and so the total x -moment of the rod is given by

$$\int_0^L x\rho(x) dx.$$

Thus the centre of the mass of the rod is given by

$$\bar{x} = \frac{\int_0^L x\rho(x) dx}{\int_0^L \rho(x) dx}.$$

Ex. The density of a 4 metre non-uniform metal beam, measured in kg/m, is given by $\rho(x) = 2\sqrt{x}$ where x is the distance from the left hand end of the beam. Find the total mass of the beam and the centre of mass.

Polar Co-ordinates:

As you saw in MATH1131, there are some interesting curves which can be easily expressed in polar co-ordinates but not so easily (if at all) in cartesian co-ordinates. Here we give the formulae for arc length and area in terms of polar co-ordinates.

We saw above that the arc length formula in parametric form was

$$\int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} dt.$$

If we have a polar curve in the form $r = r(\theta)$, we can put $x = r \cos \theta$ and $y = r \sin \theta$ and substitute to get

$$\text{arc length} = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Ex. Find the arc length of the cardioid

$$r = 1 + \sin \theta.$$

Ex. Find the length of the spiral

$$r = \theta^2$$

between $\theta = 0$ and $\theta = \pi$.

Area in Polar Co-ordinates:

To find the area, consider the wedge as shown in the diagram.

From high school you should know that the area of the sector is $\frac{1}{2}r^2\Delta\theta$. Hence, to find the area enclosed by the curve $r = f(\theta)$, for $\theta_1 \leq \theta \leq \theta_2$, we calculate

$$\frac{1}{2} \int_{\theta_1}^{\theta_2} r^2 d\theta.$$

Ex. Find the area enclosed by the cardioid
 $r = 1 - \cos \theta$.