# MATH 1231
# MATHEMATICS 1B CALCULUS.

## Section 5: - Power Series and Taylor Series.

The objective of this section is to become familiar with the theory and application of power series and Taylor series.

By the end of this section students will be familiar with:

- convergence and divergence of power and Taylor series;

- their importance;

- their uses and applications.

In particular, students will be able to solve a range of problems that involve power and Taylor series.

In general, a power series is a function of the form

$$\sum_{k=0}^{\infty} a_k(x-a)^k$$

$$= a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 + \ldots.$$

where $x$ is the variable; the $a_k$ are constants; and $a$ is some fixed number.

For example, consider the function

$$f(x) = \sum_{k=0}^{\infty} \left(\frac{x}{3}\right)^k = \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k x^k.$$

This is a power series with $a = 0$, $a_k = (1/3)^k$. What is the domain of this function? This question is equivalent to asking:

We will see that power series are a really useful way of representing some of the most important functions in mathematics, physics, chemistry and engineering.

One nice thing about power series is that they are easy to differentiate and integrate. For example, consider

$$f(x) = e^{-x^2}.$$

The above function is not easy to integrate as its integral is not an elementary function. However, if we could write $e^{-x^2}$ as a power series then we could just integrate each term of the series to get our answer.

The domain of the function $f$ represented by the power series is called the

**interval of convergence**.

Half the length of this interval is called the

**radius of convergence**.

For a given power series $\sum_{k=0}^{\infty} a_k(x-a)^k$ there are only three possibilities:

- the series converges only when $x = a$
- the series converges for all $x$
- the series converges if $|x - a| < R$ and diverges if $|x-a| > R$, for some positive number $R$.

Ex. Find the interval of convergence for the power series $\sum_{k=0}^{\infty} \frac{(x-3)^k}{3^k+2}$

Ex. Study the behaviour of $\sum_{k=0}^{\infty} k!x^k$, and $\sum_{k=0}^{\infty} \frac{x^k}{k!}$.

Ex. $\displaystyle\sum_{k=1}^{\infty} \frac{(x+1)^k \log k}{k+3}$

**Manipulation of Power Series:**
Given two power series in powers of $(x - x_0)$,

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k, \quad g(x) = \sum_{k=0}^{\infty} b_k (x - x_0)^k$$

in which both converge for $x$ in some interval of convergence, then we can add, subtract or multiply these two series and the resulting series will also converge in that interval.

More importantly,

If $f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$ for $|x - x_0| < R$ then

i) $f$ is continuous and differentiable for $|x - x_0| < R$ and

$$f'(x) = \sum_{k=0}^{\infty} k a_k (x - x_0)^{k-1};$$

ii) $f$ is integrable on $|x - x_0| < R$ and a primitive for $f$ is

$$F(x) = \sum_{k=0}^{\infty} \frac{a_k}{k+1} (x - x_0)^{k+1}.$$

Ex. Consider the power series $\sum_{k=0}^{\infty} x^k$.

Ex: Consider now, the series $f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ which we showed earlier converges for all $x$.

From these examples we can find the power series for other functions.

Ex. Find the power series for $xe^x$.

Ex. Write down the first 4 terms of the power series for $\frac{e^x}{1-x}$ and state where the series is valid.

Note that one can construct infinite series (NOT power series) in which the above results are NOT true.

Ex. Here is an example of a function defined in terms of a series which is continuous everywhere but differentiable nowhere. $f(x) = \sum_{n=1}^{\infty} (\frac{1}{2})^n \sin(4^n x)$.

As you have seen certain functions have rather nice power series. We 'stumbled' on the power series for $e^x$ by considering a special example. How can we find the power series for other functions? For example, what about $\sin x$? Does it have a simple power series and if so how do we get it? We also would like to know where such a series is valid? That is, for what values of $x$ does the series actually converge back to the given function?

The answer to this question will be given by Taylor's Theorem.

## Taylor Series:

A power series for a given function in powers of $x$ is called the **Maclaurin Series.**

We can now use this idea to obtain a formula for the Maclaurin series for a given function.

Suppose the Maclaurin series is

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ....$$

then

$$a_0 = f(0), a_1 = f'(0), a_2 = \frac{f''(0)}{2!}, a_3 = \frac{f'''(0)}{3!}, \ldots$$

This of course assumes that our function is differentiable infinitely often.

Ex. Find the Maclaurin series for $\sin x$.

Ex. Find the Maclaurin series for $\cos x$.

Ex. Find the Maclaurin series for $\log(1+x)$.

What happens if we try to find the Maclaurin series for $\log x$ in powers of $x$?

Thus, we need a more general series for a function called a Taylor series in which we try to express $f(x)$ as a polynomial in powers of $(x-x_0)$.

**Definition:** The power series

$$\sum_{k=0}^{\infty} a_k (x-x_0)^k$$

is called the Taylor Series for $f$.

The co-efficients are given by

$$a_0 = f(x_0), a_1 = f'(x_0), a_2 = \frac{f''(x_0)}{2!}, \ldots$$

(Note that if $x_0 = 0$, we just get a Maclaurin Series for $f$.)

Ex. Find the Taylor series for the function $\log(x)$ about the point $x = 1$.

We can see from the graphs (and it is true in general) that when we approximate a function $f$ by a finite number of terms of Taylor Series in powers of $x - x_0$, the closer the value of $x$ is to $x_0$ the better the approximation. We will shortly be able to measure approximately how close that will be.

Ex: Find the Taylor series for the function $\cos x$ about the point $x = \frac{\pi}{2}$, (i.e. in powers of $(x - \frac{\pi}{2})$.)

Ex: An ancient formula for computing the square root (approximately) for a given number is

$$\sqrt{a^2 + x} \approx a + \frac{x}{2a}.$$

This is simply the Maclaurin series for

$$f(x) = \sqrt{a^2 + x}.$$

**Convergence of Taylor Series:**

Two important questions arise. Firstly, exactly where does the Taylor series of a function converge? And secondly, if we truncate the Taylor series for a function at some point, how good an approximation is it?

The first of these questions requires a knowledge of Complex Analysis and is beyond the scope of the course.

To answer the second question we take the truncated Taylor series (known as the Taylor polynomial), with $(n+1)$ terms:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!}.$$

The remainder or error term, $R_n(x)$, is simply given by

$$R_n(x) = f(x) - P_n(x).$$

How big can this remainder be? Is there some way of estimating it? The following result, gives us a formula for the error involved in such an approximation.

**Taylor's Theorem:** Suppose that $f$ is (at least) $n+1$ times differentiable on the interval $(x_0 - a, x_0 + a)$. Then given $n$, for each $x \in I$, we can find a $c$ between $x_0$ and $x$ such that

$$R_n(x) = \frac{f^{(n+1)}(c)(x - x_0)^{(n+1)}}{(n+1)!}.$$

**Corollary:** If $f$ is infinitely differentiable and for each point $x$ with $|x - x_0| < R$, we have $R_n(x) \to 0$ as $n \to \infty$, then $f$ is represented by its Taylor series.

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)(x - x_0)^k}{k!}$$

Ex. Calculate the second and fourth degree Taylor approximations to $\cos x$ about the point $x_0 = 0$ and estimate how accurate they are in the range $-\frac{1}{2} \leq x \leq \frac{1}{2}$.

Ex. Estimate the error involved when $f(x) = \sqrt{x}$ is approximated by the first 3 terms of its Taylor series about $x = 9$ for $x \in (8, 10)$.

## An Application to Max. and Min.

Ex: Consider the function $f(x) = \cos(x^6)$. This clearly has a stationary point at $x = 0$ since $f'(x) = -6x^5 \sin(x^6) = 0$ at $x = 0$.

**Theorem:** Suppose $f$ is $n$ times differentiable at $x_0$, and $f'(x_0) = 0$.

If $f^{(n)}(x_0) = 0$ for $n = 2, 3, ..., k-1$ and $f^{(k)}(x_0) \neq 0$ then $f$ has

- a local minimum at $x_0$ if $k$ is even and $f^{(k)}(x_0) > 0$

- a local maximum at $x_0$ if $k$ is even and $f^{(k)}(x_0) < 0$

- an inflection point at $x_0$ if $k$ is odd.

**Indeterminate Forms:**

A limit, which presents itself as "$\frac{0}{0}$", is known as an *indeterminate form*. We met these when we looked at L'Hôpital's Rule. Using series, we can now see why L'Hôpital's Rule actually works and how to find such limits using series.

Ex: $\lim_{x \to 0} \frac{\sin x}{x}$.

Ex: $\lim_{x\to 1} \frac{\log x}{x-1}$.

$$\lim_{x\to 0} \frac{2\cos x - 2 + x^2}{x^4}.$$