

**MATH 1231
MATHEMATICS 1B CALCULUS.**

Section 4: - Convergence of Series.

The objective of this section is to get acquainted with the theory and application of series.

By the end of this section students will be familiar with:

- convergence and divergence of series;
- their importance;
- their applications.

In particular, students will be able to solve a range of problems that involve series.

A series is simply the sum of the terms of a sequence. For an infinite series we define

$$\sum_{k=j}^{\infty} a_k = \lim_{N \rightarrow \infty} \sum_{k=j}^N a_k.$$

We can think of this sum then, as the limit of a sequence of partial sums, $s_n = \sum_{k=j}^n a_k$.

Does the process make sense?

Ex.

Simple Test for Divergence:

If $a_n \not\rightarrow 0$ as $n \rightarrow \infty$ then the series $\sum_{n=j}^{\infty} a_n$ must diverge.

Be careful with this simple theorem. You cannot use it to show that a series converges. The theorem is only useful for showing that a series diverges.

Ex: The series $\sum_{n=1}^{\infty} \frac{n}{2n+3}$

The converse of this simple theorem is NOT necessarily true. For example $\sum \frac{1}{n}$ does NOT converge, despite the fact that $\frac{1}{n}$ does go to 0 as n goes to infinity.

At this stage there is only one kind of infinite series you have met and that is the infinite geometric series,

Another type of infinite series which we can deal with is the so-called telescoping series.

Ex. Find the sum $\sum_{n=1}^{\infty} \frac{1}{n^2 + n}$.

In general series are very difficult to sum, and so it is not practical to try to find a formula for the partial sums in closed form. For example, although it is true that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

but this is quite difficult to show.

Instead, we are interested in the big question:

“Does a given series converge?”

and NOT ‘What does it converge to?’

I will use the notation $\sum_{n=1}^{\infty} a_n$ for $\sum_{n=j}^{\infty} a_n$ to represent the infinite series starting at some finite value j .

Integral Test: Suppose $f(x)$ is a positive decreasing function on $[1, \infty)$, and $a_n = f(n)$ whenever n is an integer.

$$\begin{array}{c}
 a_1 \\
 a_2 \\
 a_3
 \end{array}
 \qquad
 y = f(x)$$

1 2 3

By comparing areas we see that the total area of the under-approximation is

$$\sum_{n=2}^{\infty} a_n \leq \int_1^{\infty} f(x) dx$$

and the total area of the over-approximation is

$$\sum_{n=1}^{\infty} a_n \geq \int_1^{\infty} f(x) dx.$$

Thus, we see that

- if $\int_1^{\infty} f(x) dx$ converges then $\sum_{n=1}^{\infty} a_n$ converges
- if $\int_1^{\infty} f(x) dx$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges.

Ex: Check that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Ex. Show that $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ diverges.

p -series:

Consider the series

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}.$$

Using the integral test we see that this series converges if $p > 1$ and diverges if $p \leq 1$.

Proof:

This series will be an extremely useful tool in the following test.

Comparison Test:

Suppose $0 \leq a_n \leq b_n$. That is, suppose the sequence of numbers a_n is squeezed between 0 and b_n . Then

if $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$;

if $\sum_{n=1}^{\infty} a_n$ diverges, then so does $\sum_{n=1}^{\infty} b_n$.

When making comparisons, we generally compare with the p series mentioned above.

$$\text{Ex. } \sum_{k=5}^{\infty} \frac{k^2}{k^4 + 3}$$

$$\text{Ex. } \sum_{k=5}^{\infty} \frac{k^2}{k^3 - 1}$$

The Limit Form of the Comparison Test:

This form of the Comparison test is extremely useful and allows us to rely on our intuition without having to fiddle with inequalities. It says:

Suppose

$$a_n, b_n \geq 0$$

and suppose

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} \text{ is finite AND NOT ZERO}$$

then $\sum^{\infty} a_n$ converges if and only if $\sum^{\infty} b_n$ converges.

$$\text{Ex. } \sum_{k=5}^{\infty} \frac{k^2}{k^4 + 3}$$

$$\text{Ex. } \sum_{k=5}^{\infty} \frac{7k^2 + 4k}{9k^5 - k + 1}$$

The Ratio Test:

Suppose we have a sequence a_n of positive terms, and suppose

$$\left| \frac{a_{n+1}}{a_n} \right| \rightarrow L \text{ as } n \rightarrow \infty.$$

If:

- $L < 1$ then $\sum_{n=0}^{\infty} a_n$ converges;
- $L > 1$ then $\sum_{n=0}^{\infty} a_n$ diverges;
- $L = 1$ then the test is inconclusive.

Proof of Test: When $L > 1$ we do not have $a_n \rightarrow 0$ as $n \rightarrow \infty$ so the series diverges. For $L < 1$ we have $\frac{a_{n+1}}{a_n} \leq L$ so we can show by

induction that $a_n \leq L^n a_0$ and $\sum_{n=0}^{\infty} L^n$ converges, since it is a GP with $L < 1$ and so by the comparison test, the original series converges.

We use the ratio test generally when exponentials and factorials are involved.

$$\text{Ex. } \sum_{k=1}^{\infty} \frac{k^2}{2^k}$$

$$\text{Ex. } \sum_{k=0}^{\infty} \frac{1}{k!}$$

$$\text{Ex. } \sum_{k=1}^{\infty} \frac{k!}{2^k}$$

Alternating Series:

We have seen that the series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges to infinity, i.e. the series

$$1 + \frac{1}{2} + \frac{1}{3} + \dots$$

is unbounded. Consider now, what happens if we look at the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

We let $s_n = \sum_{k=1}^n \frac{(-1)^{k+1}}{k}$ be the partial sums.

Observe that

$$s_{2m} - s_{2m-2} = \frac{1}{2m(2m-1)} > 0$$

so the sequence of even partial sums is increasing and that the even partial sums are bounded above. Thus by the Monotone Convergence Theorem, this sequence must converge.

One can similarly show that the sequence of odd partial sums also converges and so the series is convergent (in fact to $\log 2$).

Such a series is called an **alternating series**. More specifically, if a_n is a sequence of positive numbers then

$$\sum_{k=1}^{\infty} (-1)^k a_k$$

is called an alternating series. To examine the convergence of an alternating series, we begin by considering the corresponding non-alternating series

$$\sum_{k=1}^{\infty} a_k$$

which we analyse using the techniques described above.

If this series converges then the alternating series being bounded by it will also converge. We say in this case that the alternating series converges **absolutely**.

That is, a series $\sum^{\infty} a_n$ is said to converge absolutely if $\sum^{\infty} |a_n|$ converges.

As we saw in the example above, it is possible for the alternating series to converge even though the corresponding non-alternating series does not. We call this type of situation conditional convergence.

That is, a series is said to converge conditionally if the series converges but does not converge absolutely.

If the non-alternating series does not converge then we use the following test, called **Leibnitz' Test**.

Leibnitz' Test:

Suppose that a_n is a sequence of positive real numbers, and

$$i) a_1 > a_2 > a_3 > \dots,$$

i.e. $a_n > a_{n+1}$ for all n , and

$$ii) \lim_{n \rightarrow \infty} a_n = 0$$

then the alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges (conditionally).

Note that condition (i) says that the terms are monotonically decreasing. Note also that this **ONLY** works for an alternating series.

$$\text{Ex. } \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$$

$$\text{Ex. } \sum_{k=2}^{\infty} \frac{(-1)^k}{k \log k}$$

MAPLE Notes:

The following commands are relevant to the material of this chapter.

```
>sum(f,k=m..n);
```

is used to compute the sum of $f(k)$ as k goes from m to n .

```
> sum(k^2,k=1..4);
```

```
> sum(k^2,k=1..n);
```

$$\frac{1}{3}(n+1)^3 - \frac{1}{2}(n+1)^2 + \frac{1}{6}n + \frac{1}{6}$$

```
> sum(1/k^2, k=1..infinity);
```

$$\frac{\pi^2}{6}$$