

MATH 1231
MATHEMATICS 1B CALCULUS.
Section 3: - Sequences.

The objective of this section is to get acquainted with the theory and application of sequences.

By the end of this section students will be familiar with:

- limits of sequences;

- their importance;

- their applications.

In particular, students will be able to solve a range of problems that involve sequences.

A sequence is simply a function whose domain is (a subset of) the natural numbers with co-domain the real numbers.

The big question that we wish to answer, is, do the terms get closer to some finite number as we go further and further along the sequence?

That is, does

$$\lim_{n \rightarrow \infty} a_n$$

exist and, if so, then what is it?

$$\text{Ex: } a_n = \frac{1}{n}$$

$$\text{Ex: } a_n = \sin n.$$

$$\text{Ex: } a_n = \frac{n^2+1}{n}$$

$$\text{Ex: } a_n = \frac{\sin n}{n^2}$$

$$\text{Ex: } a_n = \frac{n-1}{n}.$$

Geometric Interpretation of Limits:

Ex:

If we draw little band around the line through 1, we see that eventually the crosses move into the band and stay there forever. That is, no matter how small we make the band, there is an integer N such that if we take any term further along the sequence, then it is in the band, i.e. its distance from 1 is less than the width of the band. The value of N will depend on the width of the band, the smaller the width, the larger N will have to be.

We can formalise this simple idea by saying that

This last condition simply says that the sequence is in the band.

$$\begin{array}{c} L + \epsilon \\ L \\ L - \epsilon \end{array}$$

$$N$$

Note that we are not necessarily looking for the *smallest* value of N , and in practise this might be very hard to find, we only want a value of N that works.

Ex. Prove formally that $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$

Ex. Prove formally that $\lim_{n \rightarrow \infty} \frac{(-1)^n}{(n+1)^2} = 0$

Ex. Prove formally that $\lim_{n \rightarrow \infty} \frac{3n^2 + 2n + 1}{n^2 + 5n + 1} = 3$

A similar geometric interpretation exists for sequences that diverge to $\pm\infty$.

The following simple rules are often referred to as the *algebra of limits*.

Suppose $a_n \rightarrow L$ and $b_n \rightarrow M$ as $n \rightarrow \infty$, where L and M are finite real numbers, then

Rule 1.

Rule 2.

Rule 3.

Rule 4. Suppose $f : \mathbf{R} \rightarrow \mathbf{R}$ is a function and suppose that $\lim_{x \rightarrow \infty} f(x)$ exists.

Let $\bar{f} : \mathbf{N} \rightarrow \mathbf{R}$ be the function f restricted to (some subset of) the natural numbers, i.e. $\bar{f}(n) = f(n)$. Then

$$\lim_{n \rightarrow \infty} \bar{f}(n) = \lim_{x \rightarrow \infty} f(x).$$

Rule 5. Suppose f is a continuous function, and that a_n belongs to the domain of f for each n , then $f(a_n) \rightarrow f(a)$.

Examples:

1. $\lim_{n \rightarrow \infty} \frac{4n^2 - 3n + 2}{2n^2 + 6n + 1}$

2. $\lim_{n \rightarrow \infty} \sin\left(\frac{\pi}{2} - \frac{1}{n}\right)$

3. $\lim_{n \rightarrow \infty} n \sin \frac{1}{n} =$

The following limits are standard and will be used in later work.

$$\text{A. } \lim_{n \rightarrow \infty} \frac{\ln n}{n} =$$

$$\text{B. } \lim_{n \rightarrow \infty} n^{\frac{1}{n}} =$$

$$\text{C. } \lim_{n \rightarrow \infty} x^{\frac{1}{n}} = \quad \text{for } x > 0.$$

$$\text{D. } \lim_{n \rightarrow \infty} x^n = \quad \text{for } |x| < 1.$$

Observe that **Rule 4** enables us to use L'Hôpital's rule, for example,

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

Monotone Convergence Theorem:

Suppose a_n is an increasing sequence which is bounded above. That is

$$a_0 \leq a_1 \leq a_2 \leq \dots \leq a_{n-1} \leq a_n \leq \dots \leq K$$

for all n , where K is a finite real number, then

a_n converges to some real number $L \leq K$.

(This rule is a basic property of the real numbers called *completeness*. The rational numbers are not complete.)

$$\text{Ex. } a_n = \sum_{k=0}^n \frac{1}{k!}$$

Ex. Suppose we have a sequence given recursively, for example $a_{k+1}^2 = 1 + a_k$, with starting point $a_0 = 1$. The sequence is then $1, \sqrt{2}, \sqrt{1 + \sqrt{2}}, \sqrt{1 + \sqrt{1 + \sqrt{2}}}, \dots$