MATH 1231
MATHEMATICS 1B CALCULUS.

Section 1: - Systematic Integration.

The objective of this section is to introduce students to some important techinques used in the integration of functions.

By the end of this section, students will be familiar with techniques of integration and have an understanding of how to solve a range of problems by applying the new techniques.

Students will also have an appreciation of why it is worthwhile to study integration.

1. Motivation.

What study integration?

How is it useful??

1. Basic Integrals.

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}$$

$$\int \frac{1}{\sqrt{a^2 + x^2}}\,dx = \sinh^{-1}\frac{x}{a}$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{a}\tanh^{-1}\frac{x}{a}$$

These four should be learnt carefully.

When confronted by an integral, you should always see if the integral can be 'guessed'.

Ex. Find $\int x(x^2+1)^{10}\,dx$.

Ex. Find $\displaystyle\int \frac{x}{\sqrt{x^2-4}}\,dx$.

Observe that this method ONLY works for integrals of the form $\displaystyle\int f(g(x))g'(x)\,dx$ when the derivative $g'(x)$ is present in the integrand up to a constant. You can only 'correct' the guess by multiplying or dividing by a constant.

For example, $\displaystyle\int x^4(x^3+1)^{10}\,dx$ cannot be 'guessed' since the derivative of x^3+1 is not in the integrand.

Ex. $\int \sin t \cos^5 t \, dt = -\frac{\cos^6 t}{6} + C.$

Here are some problems to practise:

1. Evaluate each of the following integrals by inspection.

DO NOT use substitution.

a. $\int xe^{2x^2}\,dx$ b. $\int x\sin(x^2)\,dx$

c. $\int x^2\cos(2x^3)\,dx$ d. $\int \frac{x}{5x^2-11}\,dx$

e. $\int \sin x\cos^3 x\,dx$ f. $\int \frac{dx}{x\,lnx}$

g. $\int \frac{x+2}{\sqrt{x^2+4x+7}}\,dx$ h. $\int x\sqrt{1+x^2}\,dx$

i. $\int x^2\sqrt{9-4x^3}\,dx$ j. $\int \frac{x^2}{\sqrt{9-4x^3}}\,dx$

k. $\int \frac{x^3}{(1+x^4)^3}\,dx$ l. $\int \frac{\sec^2 x}{\tan^4 x}\,dx$

m. $\int \frac{\cos x}{\sin^3 x}\,dx$ n. $\int e^{2x}(4+3e^{2x})^{\frac{1}{3}}\,dx$

o. $\int \frac{1}{x\,(lnx)^5}\,dx$

2. Integrate the following by parts.

a. $\int x^2 e^{-x}\, dx$ b. $\int x^3\, lnx\, dx$

c. $\int \frac{x}{\cos^2 x}\, dx$ d. $\int \frac{(lnx)^2}{x^2}\, dx$

e. $\int e^x \cos x\, dx$ f. $\int lnx\, dx$

g. $\int \tan^{-1} x\, dx$

Answers:

1a) $\frac{1}{4}e^{2x^2}$ b) $-\frac{1}{2}\cos(x^2)$ c) $\frac{1}{6}\sin(2x^3)$

d) $\frac{1}{10}\,ln|5x^2-11|$ e) $-\frac{1}{4}\cos^4 x$ f)$ln(ln\,x)$

g) $\sqrt{x^2+4x+7}$ h) $\frac{1}{3}(1+x^2)^{\frac{3}{2}}$ i) $-\frac{1}{18}(9-4x^3)^{\frac{3}{2}}$

j) $-\frac{1}{6}(9-4x^3)^{\frac{1}{2}}$ k) $-\frac{1}{8(1+x^4)^2}$ l) $\frac{-1}{3\tan^3 x}$

m) $\frac{-1}{2\sin^2 x}$ n) $\frac{1}{8}(4+3e^{2x})^{\frac{4}{3}}$ o) $\frac{-1}{4(ln\,x)^4}$

2a) $-e^{-x}(x^2+2x+2)$ b) $\frac{1}{4}x^4\,lnx-\frac{1}{16}x^4$

c) $x\tan x+ln(\cos x)$ d) $-\frac{(lnx)^2}{x}-2\frac{lnx}{x}-\frac{2}{x}$

e) $\frac{1}{2}e^x(\sin x+\cos x)$ f) $x\,lnx-x$

g) $x\tan^{-1}x-\frac{1}{2}ln(x^2+1)$

(N.B. The $+C$ has been omitted).

2. Integrals involving Trigonometric functions.

You should already be familiar with the technique to find the primitives of $\sin^2 x$ and $\cos^2 x$.

$\int \cos^2 x \, dx =$

$\int \sin^2 x \, dx =$

Integrals of the form $\int \cos^m x \sin^n x \, dx$

If m or n are odd then we can take a $\sin x$ or $\cos x$ from the odd power term use the identity $\cos^2 x + \sin^2 x = 1$ to transform the integral into one which can be guessed.

Ex. $\int \cos^4 x \sin^3 x \, dx =$

Ex. $\int \cos^5 x \sin^5 x \, dx =$

If **both** m and n are even, then the integral is slightly harder. We will see shortly a recursive method for dealing with such an integral, but for smaller values of m and n one can use the identities $\cos^2 x = \frac{1+\cos 2x}{2}$ and $\sin^2 x = \frac{1-\cos 2x}{2}$ again.

Ex. Evaluate $\int \cos^4 x \sin^2 x \, dx$

Integrals of the form $\int \cos mx \sin nx \, dx$

These type are best done using the so-called *product to sum* formulae

$\sin A \cos B =$

$\sin A \sin B =$

$\cos A \cos B =$

Ex. $\int \sin 3x \sin 2x \, dx =$

On integrals of type $\int \tan^m x \sec^n x \, dx$

If n is even, then $\sec^n x = \sec^2 x \sec^{n-2} x$ and we can substitute $\sec^2 x = 1 + \tan^2 x$ in the latter term, leading to guessable integrals.

Ex. $\int \tan^2 x \sec^4 x \, dx =$

If m is odd then take out one of the $\tan x$ terms and replace the rest using $\tan^2 x = \sec^2 x - 1$ and again we get integrals which can be guessed, recalling that the derivative of $\sec x$ is $\sec x \tan x$.

Ex. $\int \tan^3 x \sec^3 x \, dx =$

If m is even then we can use the trig identity to reduce the integrand to a power of $\sec x$ and use the reduction formula in the next section.

Note carefully the following integral for $\sec x$.

3. Revision of Integration by Parts.

Ex. For $x > 0$ find $\int x^3 ln\, x\, dx$.

Ex. $I = \int e^x \cos x\, dx =$

4. Reduction formulae

A reduction formula is a recursive formula for a family of integrals. In other words, an integral involving a power is expressed in terms of similar integrals involving a smaller power.
For example, we shall shortly show that, for $n \geq 2$,

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} x\,dx.$$

We can then apply the same formula to the second integral and reduce it down to an integral involving $\sin^{n-4} x$ and so on until the power becomes small enough to manage.

We can use a subscript notation to write the above as,

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx, \qquad I_n = \frac{n-1}{n} I_{n-2}.$$

A formula such as the one above is called a *reduction formula*.

Ex. Use the reduction formula above to find $\int_0^{\frac{\pi}{2}} \sin^7 x \, dx$.

Ex. Find the reduction formula for $I_n = \int x^n e^x \, dx$.

Ex. Find the reduction formula for $I_n = \int_0^{\frac{\pi}{2}} \sin^n x$.

The reduction formula for the integral of $\cos^n x$ is similar.

Ex. $I_n = \int_0^{\frac{\pi}{4}} \sec^n x.$

A similar method is used to get the reduction formula for the integral $\int_0^{\frac{\pi}{4}} \tan^n x.$

Ex. [Q1, Class Test 1, 2002] Let

$$I_n = \int_0^{\pi/4} \tan^n \theta \sec \theta \; d\theta.$$

Show that

$$I_n = \frac{1}{n}\left(\sqrt{2} - (n-1)I_{n-2}\right), \quad \text{for } n \geq 2.$$

Note that

$$\frac{d}{d\theta}\sec \theta = \sec \theta \tan \theta.$$

A Two parameter recurrence:

$I_{m,n} = \int_0^{\frac{\pi}{2}} \cos^m x \sin^n x \, dx$

In a similar way, one could also obtain the recurrence

$$I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}$$

for $n \geq 2$. In applying the above formulae we must reach one of $I_{1,1} = \frac{1}{2}$, $I_{1,0} = I_{0,1} = 1$ or $I_{0,0} = \frac{\pi}{2}$.

You are not expected to memorise this formula.

5. Trigonometric and Hyperbolic Substitutions.

The technique of substitution was covered in MATH1131. It involves making a 'sensible' change of variable which reduces the integral down to either a standard or a 'guessable' one.

Ex. Find $\int \log(\cos x) \tan x \, dx$, put $u = \cos x$.

Integrals involving square roots of quadratics can be worked out using the following trigonometric or hyperbolic substitutions.

$\sqrt{a^2 - x^2}$	try $x = a\sin\theta$
$\sqrt{a^2 + x^2}$	try $x = a\tan\theta$ or $x = a\sinh\theta$
$\sqrt{x^2 - a^2}$	try $x = a\sec\theta$ or $x = a\cosh\theta$

Ex. $I = \int \sqrt{1 - x^2}\, dx.$

Ex. $I = \int_0^4 x^2 \sqrt{16 - x^2}\, dx.$

Ex. $I = \int \frac{dx}{(a^2 + x^2)^{3/2}}$.

Ex. $I = \int \frac{dx}{x^2\sqrt{(x^2 - 1)}}$. Put $x = \sec\theta$

(See that this last integral can also be done using a $\cosh\theta$ substitution but it is not easy.)

6. Rational Functions and Partial Fractions.

A *rational function* is a function of the form $\dfrac{p(x)}{q(x)}$ where p and q are polynomials, for example $\dfrac{1}{x}$, $\dfrac{x^3+2}{x^6-1}$ are rational functions.

We ensure that the degree of the numerator is less than the degree of the denominator. If not then we may need to divide.

Ex. $\dfrac{x^3-x+1}{x^2+3x-1} = x-3+\dfrac{9x-2}{x^2+3x-1}$.

Ex. $\dfrac{4x-5}{2x+3} = 2-\dfrac{11}{2x+3}$.

Supposing our rational function is thus reduced, we now factorise the denominator over the real numbers, generally into linear/quadratic factors.

Factors are Linear:

Ex. $f(x) = \dfrac{2x - 1}{x^2 + 5x + 6} = \dfrac{2x - 1}{(x + 3)(x + 2)}$

In practise you use any method you can to find the co-efficients. Usually, substitute some values of x and then equate co-efficients.

Ex. $f(x) = \dfrac{2x^2 + 2x + 6}{(x-1)(x+1)(x-2)} =.$

Using the partial fraction form we can now perform the integral

$$\int \frac{2x^2 + 2x + 6}{(x-1)(x+1)(x-2)}\, dx =$$

Repeated Linear Factors:

e.g. $\dfrac{x+3}{(x+2)^3}$.

We could then evaluate $\displaystyle\int \frac{x+3}{(x+2)^3}\,dx$ to get $\dfrac{-1}{(x+2)} - \dfrac{1}{2(x+2)^2} + C$.

In general, each factor in the denominator of the form $(x-a)^k$ gives rise to an expression of the form

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots\cdots + \frac{A_k}{(x-a)^k}.$$

Ex. Evaluate $\displaystyle\int \frac{dx}{x(x+1)^3}$.

Thus $\displaystyle\int \frac{dx}{x(x+1)^3} =$

Ex. [Q2, Class Test 1, 2003] Find

$$\int \frac{8x+9}{(x-2)(x+3)^2}dx.$$

Irreducible Quadratic Factors:

Suppose the denominator of our rational function is a quadratic which does not factor over the real numbers. For example,

$$f(x) = \frac{2x + 1}{x^2 + 4x + 5}.$$

To find the integral of such a function, we *make* the numerator equal to the derivative of the denominator, leading to a logarithm and complete the square on the denominator of the remaining term leading to an inverse tangent function. *viz*

$$\int \frac{2x+1}{x^2+4x+5}\,dx =$$

(N.B. If the quadratic does have real but irrational roots, one can use this same method to avoid nasty partial fractions. The second factor will then lead to an inverse tanh.

e.g.

$$\begin{aligned} & \int \frac{2x+1}{x^2+4x-6}dx \\ = & \int \frac{2x+4}{x^2+4x-6} + \frac{3}{10-(x+2)^2}dx \\ = & \; ln|x^2+4x-6| + \frac{3}{\sqrt{10}}\tanh^{-1}\left(\frac{x+2}{\sqrt{10}}\right) + C. \end{aligned}$$

Ex. [Q1, Class Test 1, 2002] Find

$$\int \frac{x}{x^2 + 2x + 10} dx.$$

You are given that

$$\int \frac{du}{u^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C.$$

Combinations of the Rules:

One can combine these rules to cover a variety of situations.

Ex. Find $\int \frac{x+6}{x(x^2+2x+3)}\,dx$.

In general, an irreducible quadratic factor

$$x^2 + ax + b$$

in the denominator will give rise to a term of the form

$$\frac{Ax + B}{x^2 + ax + b}$$

and more generally again, a factor

$$(x^2 + ax + b)^k$$

in the denominator will give rise to terms of the form

$$\frac{A_1x + B_1}{x^2 + ax + b} + \frac{A_2x + B_2}{(x^2 + ax + b)^2} + \ldots + \frac{A_kx + B_k}{(x^2 + ax + b)^k}.$$

Ex. Find $\int \frac{x}{(x+1)^2(x^2+1)}\,dx =$

7. Further Substitutions:

The above methods using partial fractions allow us to integrate (in principle) any rational function, so we often make a change of variable in an integral that will lead us to some rational function.

Ex. Find $\int \frac{dx}{1 + x^{1/4}}$

Ex. Find $\displaystyle\int \frac{dx}{\sqrt{e^{2x} - 1}}$.

8. Half-angle substitutions:

The following substitution was once described (by Spivak) as the world's sneakiest. It is very useful for integrals involving a single term in $\sin x$ and/or $\cos x$ in the denominator.

The *Weierstrass* substitution is $t = \tan \frac{x}{2}$.

Recall, that $\sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$, and that $dx = \frac{2\,dt}{1+t^2}$.

You should know how to derive these formulae.

Who was Weierstrass??

Ex. Find $\int \frac{dx}{1 + \cos x + \sin x}$.

Ex. Show that $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x} = \frac{\pi}{3\sqrt{3}}$.

MAPLE **NOTES**

The following examples show how to use MAPLE to perform partial fraction decompositions.

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> convert(x^2/(x+2), parfrac, x)
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$$x - 2 + \frac{4}{x+2}$$

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> convert(x/(x-b)^2, parfrac, x)
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$$\frac{b}{(x-b)^2} + \frac{1}{x-b}$$