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MULTIPLE SOLUTIONS OF A BOUNDARY VALUE PROBLEM ON AN UNBOUNDED DOMAIN

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Dedicated to John Baxley on the occasion of his retirement from Wake Forest University.

ABSTRACT. We are concerned with multiple solutions of a boundary value problem on an unbounded domain. We employ degree theory coupled with the method of upper and lower solutions on compact domains. We then extend solutions to the unbounded domain and apply sequential arguments.

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1. INTRODUCTION

In this paper, we shall be interested in the existence of solutions of the boundary value problem (BVP) for the ordinary differential equation,

$$x''(t) - a(t)x(t) + f(t, x(t)) = 0, \quad 0 \le t, \tag{1.1}$$

$$x(0) = x_0, \quad x(t) \text{ bounded on } [0, \infty), \tag{1.2}$$

where x_0 is real, $f : [0, \infty) \times (-\infty, \infty) \to (-\infty, \infty)$ is continuous, $a : [0, \infty) \to (0, \infty)$ is continuous, $a(t) \ge \alpha^2 > 0$, $0 \le t$, for some $\alpha^2 > 0$.

Bebernes and Jackson [4] employed the method of subfunctions and superfunctions to study infinite interval boundary value problems for second order ordinary differential equations. More recently, Granas, Guenther, Lee and O'Regan [9], Baxley [3], and Bobisud [5], among others, have employed a variety of methods to study

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infinite interval boundary value problems for second order ordinary differential equations. Granas, Guenther, Lee and O'Regan [9] and related papers, [14], for example, employed sequential arguments in which the Arzela-Ascoli theorem is applied on compact domains. More bibliographic information is available in the recent monograph by Agarwal and O'Regan [1].

Recently, Eloe, Grimm and Mashburn [7] used the Green's function to obtain sufficient conditions for the existence of solutions of the BVP, (1.1), (1.2). It can be observed that the BVP, (1.1), (1.2), is of limit-point case [6]. Hence, it is known that a unique Green's function exists. Such Green's functions have been constructed sequentially in [8].

In this paper we shall employ degree theory to address existence of solutions. On compact domains, we shall employ two pairs of upper and lower solutions and apply degree theory in a standard way to obtain two solutions of a boundary value problem on a compact domain. We shall then apply the additivity property of degree to obtain a third solution. We then extend these solutions to the unbounded domain with sequential arguments.

The argument given here is modelled on the recent work of Henderson and Thompson [10]. In that paper they develop the machinery described in the preceding paragraph for a family of BVPs on compact domains. They construct a particularly interesting example for an autonomous system. The sufficient conditions require growth conditions on f; the conditions are similar to and they compare them to conditions one requires if one applies Leggett-Williams type theorems [17]. We refer the reader to the Avery type development of the Leggett-Williams fixed point theorems [2].

In this paper, we, too, illustrate the main theorem with an example. Thus, the primary contribution of this paper is that we obtain sufficient conditions analogous to those one expects when applying Leggett-Williams [17] or Avery [2] type methods for singular BVPs in which the singularity is due to the unbounded domain. The Leggett-Williams method requires the corresponding fixed point map is compact.

In order that the paper is self-contained, we shall provide a short discussion on degree theory in Section 2. In Section 3, we shall apply the results of Section 2 to BVPs on compact domains. We believe these results are new. In Section 4, in the main theorem, Theorem 4.1, we shall then provide sequential arguments to extend the three distinct solutions to the unbounded domain and hence obtain three distinct solutions of the BVP, (1.1), (1.2). We shall close the paper in Section 5 with an example in which we illustrate the results in Theorem 4.1.

2. BACKGROUND MATERIAL FROM DEGREE THEORY

The material briefly presented here can be found in Lloyd [12] or Schwartz [15].

Let X denote a Banach space and let $\Omega \subset X$ be an open bounded set. Assume $T : \overline{\Omega} \to X$ is a compact map and let $I : X \to X$ denote the identity map. Set $z = I - T$. Let $p \in X$ and $p \notin z(\partial\Omega)$. Let $d(z, \Omega, p)$ denote the degree of z at p relative to Ω [12]. Also, for $\lambda \in [0, 1]$, let z_λ denote the operator $I - \lambda T$.

Theorem 2.1. *If $d(z, \Omega, p) \neq 0$, then the equation $z(x) = p$ has at least one solution, $x \in \Omega$.*

Theorem 2.2. *Let z, Ω, p be as above and define $z_\lambda = I - \lambda T$, $0 \le \lambda \le 1$, with $d(z_\lambda, \Omega, p)$ defined for all $\lambda \in [0, 1]$. Then $d(z_0, \Omega, p) = d(z_1, \Omega, p)$.*

Since $d(z_0, \Omega, p)$ can be readily calculated we have the following theorem.

Theorem 2.3. *Let z, Ω, p be as above. Then*

$$d(z, \Omega, p) = \begin{cases} 1, & p \in \Omega, \\ 0, & p \notin \Omega. \end{cases}$$

The last two results are crucial to the strategy to obtain three solutions.

Theorem 2.4. *Let z, Ω, p be as above. Assume $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ where each Ω_i is open and nonempty and the family $\{\Omega_i\}$ is disjoint. Then*

$$d(z, \Omega, p) = d(z, \Omega_1, p) + d(z, \Omega_2, p) + d(z, \Omega_3, p).$$

Theorem 2.5. *let $K \subset \overline{\Omega}$, assume K is closed, and assume $p \notin z(K)$. Then $d(z, \Omega, p) = d(z, \Omega \setminus K, p)$.*

3. AN ASSOCIATED BOUNDARY VALUE PROBLEM ON BOUNDED DOMAINS

We first address a right focal [13] boundary value problem on a bounded domain. Let $B_0 > 0$ and let $b > B_0$. We shall first consider the boundary value problem

$$x''(t) - a(t)x(t) + f(t, x(t)) = 0, \quad 0 < t < b, \tag{3.1}$$

$$x(0) = x_0, \quad x'(b) = 0. \tag{3.2}$$

Some of the work here is modelled by the work of Eloe, Grimm and Mashburn [7].

Definition 3.1. *We say the α_b is a C^0-lower solution of the BVP, (3.1), (3.2), if $\alpha_b \in C^0[0,b]$, $\alpha_b'(b)$ exists, and for each $t \in (0,b)$ there exists an open interval I_t such that $t \in I_t \subset (0,b)$ and an $\alpha_t \in C^2(I_t)$ such that*

$$\alpha_t(s) \le \alpha_b(s), \quad s \in I_t, \quad \alpha_t(t) = \alpha_b(t),$$

$$\alpha_t''(s) - a(s)\alpha_t(s) + f(s, \alpha_t(s)) \ge 0, \quad s \in I_t, \tag{3.3}$$

$$\alpha_b(0) \le x_0, \quad \alpha_b'(b) < 0. \tag{3.4}$$

We say the β_b is a C^0-upper solution of the BVP, (3.1), (3.2), if $\beta_b \in C^0[0,b]$, $\beta_b'(b)$ exists, and for each $t \in (0,b)$ there exists an open interval I_t such that $t \in I_t \subset (0,b)$ and $\beta_t \in C^2(I_t)$ such that

$$\beta_t(s) \ge \beta_b(s), \quad s \in I_t, \quad \beta_t(t) = \beta_b(t),$$

$$\beta_t''(s) - a(s)\beta_t(s) + f(s, \beta_t(s)) \le 0, \quad s \in I_t,$$

$$\beta_b(0) \ge x_0, \quad \beta_b'(b) > 0.$$

We will say α_b is a strict C^0-lower solution of the BVP, (3.1), (3.2), if the inequality in (3.3) is strict for each $t \in (0,b)$. A strict C^0-upper solution is defined similarly.

Let $G(b;t,s)$ denote the Green's function for the right focal problem, $-(x''(t) - a(t)x(t)) = 0$, $0 < t < b$, with the boundary conditions, $x(0) = 0, x'(b) = 0$. Let $p_b(t)$ denote the solution of the BVP, $x''(t) - a(t)x(t) = 0$, $0 < t < b$, with the boundary conditions, $x(0) = x_0, x'(b) = 0$. The uniqueness and existence of p is shown in [7]. Define the operator $T_b : C[0,b] \to C[0,b]$ by

$$T_b x_b(t) = p_b(t) + \int_0^b G(b;t,s) f(s, x_b(s)) ds. \tag{3.5}$$

Proving existence of a solution of the BVP, (3.1), (3.2), is equivalent to proving the existence of a fixed point of T_b.

Theorem 3.2. *Assume f is bounded on $(0,\infty) \times (-\infty,\infty)$. Then $d(I - T_b, \Omega_b, 0) = 1$; in particular, T_b has a fixed point in $C[0,b]$.*

Proof. Let

$$N = \sup |f(t,x)|, \quad 0 < t, \quad -\infty < x < \infty,$$

$$G_b = \max_{[0,b]\times[0,b]} |G(b;t,s)|,$$

$$||p_b||_b = \max_{0 \le t \le b} |p_b(t)|,$$

$$\Omega_b = \{x_b \in C[0,b] : ||x_b|| < ||p_b||_b + G_b N + 1\}.$$

Consider the family of mappings

$$z_\lambda = I - \lambda T_b, \quad \lambda \in [0,1],$$

where T_b is defined in (3.5). It follows by straight-forward applications of Theorems 2.2 and 2.3 that $d(I - T_b, \Omega_b, 0) = 1$. This completes the proof of Theorem 3.2. □

Theorem 3.3. *Let α_b, β_b denote C^0-lower and C^0-upper solutions, respectively, of the BVP, (3.1), (3.2). Assume $\alpha_b(t) \le \beta_b(t)$, $0 \le t \le b$. Assume f satisfies*

$$f(t, \beta_b(t)) < f(t,x), \quad \textit{for all} \quad \beta_b(t) < x, \tag{3.6}$$

$$f(t, \alpha_b(t)) > f(t,x), \quad \textit{for all} \quad \alpha_b(t) > x. \tag{3.7}$$

If $x_b(t)$ is a solution of BVP, (3.1), (3.2), then

$$\alpha_b(t) \le x_b(t) \le \beta_b(t), \quad t \in [0,b].$$

Proof. The proof is by contradiction and standard [11]. Assume for the sake of contradiction that $x_b(t) \le \beta_b(t)$, $t \in [0,b]$ is false. Then $x_b - \beta_b$ has a positive maximum in $[0,b]$. If the positive maximum occurs at $t_0 \in (0,b)$ then $x_b - \beta_{t_0}$ attains a positive maximum at t_0. Then (3.6) gives the contradiction

$$(x_b - \beta_{t_0})''(t_0) \ge f(t_0, x_b(t_0)) - f(t_0, \beta_{t_0}(t_0)) > 0.$$

Thus, the positive maximum does not occur in $(0,b)$. The positive maximum does not occur at $t_0 = 0$, since $(x_b - \beta_b)(0) \le 0$. So the only other possibility is that the positive maximum occurs as $t_0 = b$. Then $(x_b - \beta_b)'(b) \ge 0$. But $x_b'(b) = 0$ and $\beta_b'(b) > 0$ contradicts this inequality. This completes the proof of Theorem 3.3. □

We have need for stronger inequalities which are given in the next theorem.

Theorem 3.4. *Let α_b, β_b denote strict C^0-lower and strict C^0-upper solutions, respectively, of the BVP, (3.1), (3.2). Assume $\alpha_b(t) \le \beta_b(t)$, $0 \le t \le b$. Assume f satisfies*

$$f(t, \beta_b(t)) < f(t,x), \quad \textit{for all} \quad \beta_b(t) < x, \tag{3.8}$$

$$f(t, \alpha_b(t)) > f(t,x), \quad \textit{for all} \quad \alpha_b(t) > x. \tag{3.9}$$

If $x(t)$ is a solution of BVP, (3.1), (3.2), then

$$\alpha_b(t) < x(t) < \beta_b(t), \quad t \in (0,b].$$

Proof. The proof is similar to the proof of Theorem 3.3. The strict lower and upper solutions give strict inequalities in the conclusion. □

At this point we change notation for more clear exposition. Let $\alpha_b(t)$ be denoted by $\alpha(b;t)$. Let $\beta_b(t)$ be denoted by $\beta(b;t)$.

Theorem 3.5. *Assume* $f : [0, b] \times (-\infty, \infty) \to (-\infty, \infty)$ *is continuous and assume there exist strict* C^0*-lower and strict* C^0*-upper solutions of BVP, (3.1), (3.2), respectively,* $\alpha_2(b; t)$ *and* $\beta_1(b; t)$*, and lower and upper solutions of BVP, (3.1), (3.2), (not necessarily strict) respectively,* $\alpha_1(b; t)$ *and* $\beta_2(b; t)$*, all satisfying*

$$\alpha_1(b; t) \le \alpha_2(b; t) \le \beta_2(b; t), \quad 0 \le t \le b, \tag{3.10}$$

$$\alpha_1(b; t) \le \beta_1(b; t) \le \beta_2(b; t), \quad 0 \le t \le b, \tag{3.11}$$

$$\alpha_2(b; t) \not\le \beta_1(b; t), \quad 0 \le t \le b. \tag{3.12}$$

Then the BVP, (3.1), (3.2) has at least three solutions, $x_1(b; t), x_2(b; t), x_3(b; t)$ *satisfying*

$$\alpha_1(b; t) \le x_1(b; t) \le \beta_1(b; t),\ \alpha_2(b; t) \le x_2(b; t) \le \beta_2(b; t), \quad 0 \le t \le b,$$

$$x_3(b; t) \not\le \beta_1(b; t),\ \alpha_2(b; t) \not\le x_3(b; t), \quad 0 \le t \le b.$$

Proof. We modify the BVP, (3.1), (3.2), in a standard way [11]. Define a bounded function, F_1, by

$$F_1(t, x) = \begin{cases} f(t, \beta_2(b; t)) + \frac{x - \beta_2(b;t)}{1 + |x - \beta_2(b;t)|}, & \beta_2(b; t) \le x, \\ f(t, x), & \alpha_1(b; t) \le x \le \beta_2(b; t), \\ f(t, \alpha_1(b; t)) + \frac{x - \alpha_1(b;t)}{1 + |x - \alpha_1(b;t)|}, & x \le \alpha_1(b; t). \end{cases}$$

Clearly F_1 is bounded and F_1 satisfies (3.6) and (3.7). Consider the modified BVP,

$$x''(t) - a(t)x(t) + F_1(t, x(t)) = 0, \quad 0 < t < b, \tag{3.13}$$

satisfying boundary conditions (3.2).

Apply Theorem 3.2 and obtain a solution, x, of the BVP, (3.13), (3.2). Since $F_1(t, \alpha_1(t)) = f(t, \alpha_1(t))$ and $F_1(t, \beta_2(t)) = f(t, \beta_2(t))$ it follows that α_1 and β_2 serve as lower and upper solutions, respectively, of the BVP, (3.13), (3.2), as well. Apply Theorem 3.3 and note that

$$\alpha_1(t) \le x(t) \le \beta_2(t), \quad 0 \le t \le b,$$

and so, x is a solution of the original BVP, (3.1), (3.2).

Let N denote an upper bound on F_1 and define Ω_b as in Theorem 3.2. Define

$$\Omega_1 = \{x \in \Omega_b : x(t) > \alpha_2(t), 0 \le t \le b\},$$

$$\Omega_2 = \{x \in \Omega_b : x(t) < \beta_1(t), 0 \le t \le b\}.$$

Since $\alpha_2 \not\le \beta_1$, $\alpha_2 > -(||p_b||_b + G_b N + 1)$ and $\beta_1 < (||p_b||_b + G_b N + 1)$, it follows that each of the sets, Ω_1, Ω_2, $\overline{\Omega}_1 \cap \overline{\Omega}_2$, and $\Omega \setminus \overline{\Omega_1 \cup \Omega_2}$ are all non-empty.

The sets Ω_1, Ω_2, and $\Omega \setminus \overline{\Omega_1 \cup \Omega_2}$ are mutually disjoint. Moreover, since α_2 and β_1 are strict lower and upper solutions, respectively, of the BVP, (3.1), (3.2), there are no solutions $x \in \partial\Omega_1 \cup \partial\Omega_2$ of the BVP, (3.1), (3.2), by Theorem 3.4.

To apply Theorem 3.2 define

$$T_1 x(t) = p_b(t) + \int_0^b G(b;t,s)F_1(s,x(s))ds, \quad 0 \le t \le b.$$

Apply Theorems 2.4 and 2.5 to obtain

$$\begin{aligned} 1 =& d(I - T_1, \Omega, 0) \\ =& d(I - T_1, \Omega_1, 0) + d(I - T_1, \Omega_2, 0) + d(I - T_1, \Omega \setminus \overline{\Omega_1 \cup \Omega_2}, 0). \end{aligned}$$

Consider a second modified BVP,

$$x''(t) - a(t)x(t) + F_2(t, x(t)) = 0, \quad 0 < t < b, \tag{3.14}$$

satisfying boundary conditions (3.2) where

$$F_2(t,x) = \begin{cases} f(t,\beta_2(t)) + \frac{x-\beta_2(t)}{1+|x-\beta_2(t)|}, & \beta_2(t) \le x, \\ f(t,x), & \alpha_2(t) \le x \le \beta_2(t), \\ f(t,\alpha_2(t)) + \frac{x-\alpha_2(t)}{1+|x-\alpha_2(t)|}, & x \le \alpha_2(t). \end{cases}$$

Define

$$T_2 x(t) = p_b(t) + \int_0^b G(b;t,s)F_2(s,x(s))ds, \quad 0 \le t \le b.$$

It follows by Theorem 3.2 that $d(I - T_2, \Omega, 0) = 1$.

Apply Theorem 2.4 and the contrapositive of Theorem 2.1 to see that

$$d(I - T_2, \Omega \setminus \overline{\Omega}_1, 0) = 0.$$

Thus,

$$\begin{aligned} d(I - T_1, \Omega_1, 0) &= d(I - T_2, \Omega_1, 0) \\ &= d(I - T_2, \Omega_1, 0) + d(I - T_2, \Omega \setminus \overline{\Omega}_1, 0) \\ &= d(I - T_2, \Omega, 0) = 1. \end{aligned}$$

Similarly, $d(I - T_1, \Omega_2, 0) = 1$. Thus, apply Theorem 2.4 to obtain

$$d(I - T_1, \Omega \setminus \overline{\Omega_1 \cup \Omega_2}, 0), 0) = -1.$$

It now follows by Theorem 2.1 that there exist three solutions $x_{b1} \in \Omega_{b1}$, $x_{b2} \in \Omega_{b2}$, and $x_{b3} \in \Omega_b \setminus \overline{\Omega_{b1} \cup \Omega_{b2}}$, of the BVP, (3.1), (3.2). This completes the proof of Theorem 3.5. □

4. EXTENDING THREE SOLUTIONS TO THE UNBOUNDED DOMAIN

In this section, we show that limiting arguments can be used to construct three distinct solutions of the singular BVP, (1.1), (1.2). The technical details are provided in [7] and we only outline those details here.

Theorem 4.1. *Assume* $f : [0,\infty)\times(-\infty,\infty) \to (-\infty,\infty)$ *is continuous. Let* $B > 0$. *Assume the existence of* α_2 *and* β_1 *such that, for each* $b > B$, α_2 *is a strict* C^0*-lower solutions and* β_1 *is a strict* C^0*-upper solutions of BVP, (3.1), (3.2), respectively; assume the existence of* α_1 *and* β_2 *such that, for each* $b > B$, α_1 *is a* C^0*-lower solutions and* β_2 *is a* C^0*-upper solutions of BVP, (3.1), (3.2), respectively. Assume*

$$\alpha_1(t) \le \alpha_2(t) \le \beta_2(t), \quad 0 \le t,$$

$$\alpha_1(t) \le \beta_1(t) \le \beta_2(t), \quad 0 \le t,$$

$$\alpha_2(t) \not\le \beta_1(t), \quad 0 \le t.$$

Then the BVP, (1.1), (1.2) has at least three solutions, $x_1(t), x_2(t), x_3(t)$ *satisfying*

$$\alpha_1(t) \le x_1(t) \le \beta_1(t), \alpha_2(t) \le x_2(t) \le \beta_2(t), \quad 0 \le t,$$

$$x_3(t) \not\le \beta_1(t), \alpha_2(t) \not\le x_3(t), \quad 0 \le t.$$

Outline of Proof. Let $\{b_i\}$ denote an unbounded increasing sequence of reals. Assume without loss of generality that $B \le b_1$. For each b_i, apply Theorem 3.5 and obtain the three distinct functions, $x_1(b_i;t), x_2(b_i;t), x_3(b_i;t)$. Define for each $j = 1,2,3$,

$$u_j(b_i;t) = \begin{cases} x_j(b_i;t), & 0 \le t \le b_i, \\ x_j(b_i;b), & b_i < t. \end{cases}$$

Consider the family of functions, $\{u_j(b_i;t)|_{[0,b_l]}\}$ where we mean the restriction of $u_j(b_i;t)$ to the interval $[0,b_l]$ for some l. There is a subsequence (for details see [7]) that converges in $C^2[0,b_l]$ to say $w_j(b_l;t)$. Perform this construction inductively for $l = 1,2,\ldots$. By the inductive construction,

$$w_j(b_{l+1};t) = w_j(b_l;t), \quad 0 \le t \le b_l.$$

Define

$$x_j(t) = \cup_l w_j(b_l;t), \quad j = 1,2,3.$$

Then $x_1(t), x_2(t)$ and $x_3(t)$ denote the three distinct solutions of the BVP, (1.1), (1.2), and the outline of the details is complete.

5. AN EXAMPLE

We show that the autonomous boundary value problem,

$$x(t)'' - x(t) + f(x(t)) = 0, \quad 0 \le t, \tag{5.1}$$

$$x(0) = 2, \quad x(t) \text{ bounded on } [0, \infty), \tag{5.2}$$

has three positive solutions under certain growth conditions on f.

Theorem 5.1. *Let $f : [0, +\infty) \to [0, +\infty)$ be such that $f(x) > 0$ if $x > 0$. Assume there exist real numbers a and c such that*

(i) $0 < a < \cosh(1), 2\cosh(2) - 1 < c$,
(ii) $f(x) < 2 + a$ *for* $x \in [0, a]$,
(iii) $f(x) > 2\cosh(2)$ *for* $x \in \big[\cosh(1), 2\cosh(2) - 1\big]$,
(iv) $f(x) \le 2 + c$ *for* $x \in [0, c]$.

Then the BVP, (5.1), (5.2), *has three positive solutions x_1, x_2, and x_3 satisfying*

$$e^{-t} \le x_1(t) \le 2 + a(1 - e^{-t}), \quad \alpha_2(t) \le x_2(t) \le 2 + c(1 - e^{-t})$$

and

$$x_3(t) \not\le 2 + a(1 - e^{-t}), \quad \alpha_2(t) \not\le x_3(t),$$

for all $t > 0$, where α_2 is given by

$$\alpha_2(t) = \begin{cases} \cosh(t), & \text{for all } 0 \le t \le 2, \\ 2\cosh(2) - \cosh(t - 4), & \text{for all } 2 < t < 6, \\ \cosh(2)e^{6-t}, & \text{for all } 6 \le t. \end{cases}$$

Proof. Set $B > 6$ and let $b > B$. It is clear that $\beta_1(t) = 2 + a(1 - e^{-t})$ and $\beta_2(t) = 2 + c(1 - e^{-t})$ are upper solutions for (5.1), (5.2) with β_1 being a strict upper solution. Also, it is clear $\alpha_1(t) = e^{-t}$ is a lower solution for (5.1), (5.2).

We show that $\alpha_2(t)$ is a strict C^0-lower solution. First note that $\alpha_2(4) = 2\cosh(2) - 1 > \beta_1(4)$, and hence $\alpha_2(t) \not\le \beta_1(t)$, for $0 \le t \le b$. Since $\alpha_2(2^+) = \alpha_2(2^-)$ and $\alpha_2(6^+) = \alpha_2(6^-)$, then $\alpha_2 \in C^0[0, b]$. Furthermore, for $t \in (0, 2)$ and $t \in (6, +\infty)$ it is easy to see that $\alpha_2(t)'' - \alpha_2(t) + f(\alpha_2(t)) = f(\alpha_2(t)) > 0$. If $t \in (2, 6)$, then $\alpha_2(t) > \cosh(1)$, and so $\alpha_2(t)'' - \alpha_2(t) + f(\alpha_2(t)) = -2\cosh(2) + f(\alpha_2(t)) > 0$. Now let $t_0 = 2$ and let $I_{t_0} = (2 - \varepsilon, 2 + \varepsilon)$, where $\varepsilon > 0$ is chosen so that, for all $t \in I_{t_0}$, we have $2\cosh(2) - \cosh(t - 4) > \cosh(1)$ and $2 - \varepsilon > 1$. Then $a_2(t) \ge 2\cosh(2) - \cosh(t - 4) := \alpha_{2,t_0}(t)$, and for all $t \in I_{t_0}$,

$$\alpha''_{2,t_0}(t) - \alpha_{2,t_0}(t) + f(\alpha_{2,t_0}) = -2\cosh(2) + f(\alpha_{2,t_0}) > 0.$$

A similar argument holds at $t_0 = 6$ and so α_2 is a C^0-lower solution. An application of Theorem 4.1 yields the existence of the three solutions and the proof is complete. □

Remark: The function

$$f(x) = \begin{cases} 2x, & 0 \leq x \leq 2, \\ 2 + \frac{6}{\cosh(1)-1}(x-1), & 2 \leq x \leq \cosh(1), \\ 8, & \cosh(1) \leq x \leq 2\cosh(2)-1, \\ 8 + \frac{2}{9-2\cosh(2)}, & 2\cosh(2)-1 \leq x \leq 8, \\ 10, & x \geq 8, \end{cases}$$

satisfies conditions (ii) - (iv) of Theorem 5.1 when $a = 1$ and $c = 8$.

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