

# Solvability of Discrete Neumann Boundary Value Problems

D. R. Anderson♣, I. Rachůnková♡ and C. C. Tisdell♠

September 17, 2006

1

♣ Department of Mathematics and Computer Science  
Concordia College,  
901 8th Street, Moorhead MN 56562, USA  
email: andersod@cord.edu

♡ Department of Mathematics  
Palacký University  
771 46 Olomouc, Czech Republic  
email: rachunko@risc.upol.cz

♠ School of Mathematics and Statistics  
The University of New South Wales  
Sydney 2052, Australia  
email: cct@maths.unsw.edu.au

## Abstract

In this article we gain solvability to a nonlinear, second-order difference equation with discrete Neumann boundary conditions. Our methods involve new inequalities on the right-hand side of the difference equation and Schaefer's theorem in the finite-dimensional space setting.

Running Head: Discrete BVPs  
AMS Subject Code: 39A12, 34B15  
Corresponding Author: C C Tisdell

Keywords and Phrases: discrete Neumann boundary value problem; existence of solutions; Schaefer's Theorem; difference equation.

---

<sup>1</sup>This research was supported by the Australian Research Council's Discovery Projects DP0450752 and the Council of Czech Government MSM 6198959214

# 1 Introduction

This paper investigates the following discrete Neumann boundary value problem (BVP)

$$(1.1) \quad \nabla \Delta y(k) = f(k, y(k), \Delta y(k)), \quad k = 1, \dots, n-1;$$

$$(1.2) \quad \Delta y(0) = 0 = \Delta y(n);$$

where:  $f$  is a continuous, scalar-valued function;  $n \geq 2$ ; and the differences are given by:

$$\Delta y(k) := \begin{cases} y(k+1) - y(k), & \text{for } k = 0, \dots, n-1, \\ 0, & \text{for } k = n; \end{cases}$$

$$\nabla y(k) := \begin{cases} y(k) - y(k-1), & \text{for } k = 1, \dots, n, \\ 0, & \text{for } k = 0; \end{cases}$$

$$\nabla \Delta y(k) := \begin{cases} y(k+1) - 2y(k) + y(k-1), & \text{for } k = 1, \dots, n-1, \\ 0, & \text{for } k = 0 \text{ or } k = n. \end{cases}$$

This paper addresses a question of interest regarding the discrete BVP (1.1), (1.2):

- Under what conditions does the discrete BVP (1.1), (1.2) have at least one solution?

Particular significance in the above question lies in the fact that strange and interesting distinctions can occur between the theory of differential equations and the theory of difference equations. For example, properties such as existence, uniqueness and multiplicity of solutions may not be shared between the theory of differential equations and the theory of difference equations [1, p.520], even though the right hand side of the equations under consideration may be the same. Moreover, when investigating difference equations, as opposed to differential equations, basic ideas from calculus are not necessarily available to use, such as the intermediate value theorem; the mean value theorem and Rolle's theorem. Thus, new challenges are faced and innovation is required.

The paper is organised as follows.

Section 2 contains the main results of the paper. There, some sufficient conditions are presented, in terms of difference inequalities involving  $f$ , such that (1.1), (1.2) will admit at least one solution. The main ideas of the proof involve *a priori* bounds on solutions to a certain family of problems, and also involves Schaefer's Theorem [8, Theorem 4.4.10] in the finite-dimensional space setting.

In Section 3 some examples are presented to illustrate how to apply the new theory.

For recent and classical results on difference equations and their comparison with differential equations, including existence, uniqueness and spurious solutions, the reader is referred to: [1]-[7], [9]-[16].

A solution to problem (1.1) is a vector  $\mathbf{y} = (y(0), \dots, y(n)) \in \mathbb{R}^{n+1}$  satisfying (1.1) for  $k = 1, \dots, n-1$ .

We will need the following identity in the proof of our main theorem, obtained from the discrete product rule. If  $r(t) := [y(t)]^2$ ,  $t \in \mathbb{Z}$  then

$$(1.3) \quad \nabla \Delta r(t) = 2y(t) \nabla \Delta y(t) + [\Delta y(t)]^2 + [\nabla y(t)]^2.$$

## 2 Main Results

In this section we present and prove the main results of the paper. The main ideas involve new difference inequalities (on  $f$ ) and Schaefer's Theorem [8, Theorem 4.4.10] in the finite-dimensional space setting.

**Theorem 2.1** *Let  $f$  be continuous. If there exist non-negative constants  $\alpha$  and  $K$  such that*

$$(2.1) \quad |f(t, p, q) - p| \leq \alpha [2pf(t, p, q) + q^2] + K, \quad \forall (t, p, q) \in \{1, \dots, n-1\} \times \mathbb{R} \times \mathbb{R};$$

*then the discrete BVP (1.1), (1.2) has at least one solution.*

**Proof** We consider the following discrete BVP that is equivalent to (1.1), (1.2), namely

$$(2.2) \quad \nabla \Delta y(k) - y(k) = f(k, y(k), \Delta y(k)) - y(k), \quad k = 1, \dots, n-1;$$

$$(2.3) \quad \Delta y(0) = 0 = \Delta y(n).$$

We will prove that (2.2), (2.3) has at least one solution and thus, so will (1.1), (1.2). We may reformulate (2.2), (2.3) as an equivalent summation equation, namely

$$y(k) = \sum_{i=1}^{n-1} G(k, i) [f(i, y(i), \Delta y(i)) - y(i)], \quad k = 0, \dots, n,$$

where  $G$  is the unique, continuous Green's function associated with the discrete BVP

$$\begin{aligned} \nabla \Delta y(k) - y(k) &= 0, \quad k = 1, \dots, n-1; \\ \Delta y(0) &= 0 = \Delta y(n). \end{aligned}$$

Introduce the operator (defined componentwise below)  $\mathbf{T} : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$  by

$$(2.4) \quad T_k(\mathbf{y}) = \sum_{i=1}^{n-1} G(k, i) [f(i, y(i), \Delta y(i)) - y(i)], \quad k = 0, \dots, n,$$

Thus, we want to show that there exists at least on  $\mathbf{y} \in \mathbb{R}^{n+1}$  such that

$$(2.5) \quad \mathbf{y} = \mathbf{T}\mathbf{y}.$$

To do this, we will use Schaefer's Theorem [8, Theorem 4.4.10] in the finite-dimensional space setting.

Since  $f$  and  $G$  are continuous, see that  $\mathbf{T}$  is a continuous map (and thus compactness of  $\mathbf{T}$  in the finite-dimensional space setting is guaranteed).

It remains to show that all possible solutions to

$$(2.6) \quad \mathbf{y} = \lambda \mathbf{T}\mathbf{y}, \quad \lambda \in [0, 1].$$

are bounded *a priori*, with the bound being independent of  $\lambda$ . With this in mind, let  $\mathbf{y}$  be a solution to (2.6) and denote

$$G_0 := \max\{|G(k, s)| : (k, s) \in [0, n]^2\}.$$

For each  $k = 0, \dots, n$  we have

$$\begin{aligned} |y(k)| &= \lambda |T_k y(k)| \\ &\leq G_0 \sum_{i=1}^{n-1} \lambda |f(i, y(i), \Delta y(i)) - y(i)| \\ &\leq G_0 \sum_{i=1}^{n-1} \alpha [2y(i)\lambda f(i, y(i), \Delta y(i)) + \lambda[\Delta y(i)]^2] + \lambda K, \quad \text{from (2.1)} \\ &\leq G_0 \sum_{i=1}^{n-1} \alpha [2y(i)\lambda f(i, y(i), \Delta y(i)) + 2(1 - \lambda)[y(i)]^2 + [\Delta y(i)]^2 + [\nabla y(i)]^2] + K \\ &= G_0 \sum_{i=1}^{n-1} \alpha [2y(i)\nabla \Delta y(i) + [\Delta y(i)]^2 + [\nabla y(i)]^2] + K \\ &= G_0 \sum_{i=1}^{n-1} \alpha \nabla \Delta r(i) + K, \quad \text{from (1.3)} \\ &= G_0 (\alpha [\nabla r(n) - \nabla r(1)] + Kn) \\ &= G_0 (\alpha [(y(n) + y(n-1))\nabla y(n) - (y(1) - y(0))\nabla y(1)] + Kn) \\ &= G_0 Kn, \quad \text{from (1.2)}. \end{aligned}$$

Hence we see that all solutions to the family (2.6) are bounded *a priori*, with the bound being independent of  $\lambda$ . Schaefer's Theorem applies to  $\mathbf{T}$ , yielding the existence of at least one fixed point.  $\square$

The following corollary easily follows to Theorem 2.1 when  $f(t, p, q) - p$  is bounded.

**Corollary 2.2** *If  $f(t, p, q) - p$  is continuous and bounded on  $\{1, \dots, n-1\} \times \mathbb{R}^2$  then the discrete BVP (1.1), (1.2) has at least one solution.*

**Proof** The proof follows by choosing  $\alpha = 0$  and  $K$  to be larger than the bound on  $f(t, p, q) - p$ . Thus, the conditions of Theorem 2.1 are satisfied and the result follows.  $\square$

If the right-hand side of (1.1) does not depend on  $\Delta y(k)$  then we obtain the following discrete Neumann BVP

$$(2.7) \quad \nabla \Delta y(k) = f(k, y(k)), \quad k = 1, \dots, n-1;$$

$$(2.8) \quad \Delta y(0) = 0 = \Delta y(n);$$

and the following important corollary to Theorem 2.1 follows.

**Corollary 2.3** *Let  $f$  be continuous. If there exist non-negative constants  $\alpha$  and  $K$  such that*

$$(2.9) \quad |f(t, p) - p| \leq 2\alpha p f(t, p) + K, \quad \forall (t, p) \in \{1, \dots, n-1\} \times \mathbb{R};$$

*then the discrete BVP (2.7), (2.8) has at least one solution.*

**Proof** If (2.9) holds then it is easy to see that

$$|f(t, p) - p| \leq \alpha[2pf(t, p) + q^2] + K, \quad \forall (t, p, q) \in \{1, \dots, n-1\} \times \mathbb{R} \times \mathbb{R};$$

and so the conditions of Theorem 2.1 hold with the result following from there.  $\square$

### 3 Examples

In this section examples are discussed to highlight how to apply the theory of Section 2.

**Example 3.1** Consider the discrete Neumann BVP (2.7), (2.8) where  $f$  is given by

$$f(t, p) = p^5 + p + t, \quad t = 1, \dots, 9,$$

(and  $n = 10$ ). We claim that for the above  $f$ , the discrete BVP (2.7), (2.8) has at least one solution.

**Proof** We want to show that there exist non-negative constants  $\alpha$  and  $K$  such that (2.9) holds.

See that, for  $(t, p) \in \{1, \dots, 9\} \times \mathbb{R}$  we have

$$|f(t, p) - p| \leq |p^5| + 9.$$

For  $\alpha$  and  $K$  to be chosen below, for  $(t, p) \in \{1, \dots, 9\} \times \mathbb{R}$  consider

$$\begin{aligned} 2\alpha p f(t, p) + K &= 2\alpha[p^6 + p^2 + pt] + K \\ &= (p^6 + 1) + [p^2 + pt + 49], \quad \text{for } \alpha = 1/2, K = 50 \\ &\geq (|p^5|) + [9] \geq |f(t, p) - p| \end{aligned}$$

and the result follows from Corollary 2.3.  $\square$

The second example provides the function  $f$  depending also on  $q$ .

**Example 3.2** Consider the discrete Neumann BVP (1.1), (1.2) where  $f$  is given by

$$f(t, p, q) = \frac{3 - pt}{p^2 + 1} + p + q^2(1 + \sin(\pi q/2)) \operatorname{sign} p, \quad t = 1, \dots, 99,$$

(and  $n = 100$ ). We claim that for the above  $f$ , the discrete BVP (1.1), (1.2) has at least one solution. Note that  $f$  is not monotone in  $q$  for fixed  $t$  and  $p$  and that  $f$  can change its sign with respect to  $t$  for fixed  $p$  and  $q$ .

**Proof** Denote

$$P_0 = \max \left\{ \left| \frac{3p - p^2t}{p^2 + 1} \right| : t \in \{1, \dots, 99\}, p \in \mathbb{R} \right\},$$

$$P_1 = \max \left\{ \left| \frac{3 - pt}{p^2 + 1} \right| : t \in \{1, \dots, 99\}, p \in \mathbb{R} \right\},$$

$K = 4P_0 + P_1$ ,  $\alpha = 2$ . Then

$$\begin{aligned} |f(t, p, q) - p| &\leq \left| \frac{3 - pt}{p^2 + 1} \right| + 2q^2 \\ &\leq K + 4 \frac{3p - p^2t}{p^2 + 1} + 2q^2 + 4p^2 + 4|p|q^2(1 + \sin(\pi q/2)) \\ &= \alpha[2pf(t, p, q) + q^2] + K \end{aligned}$$

for  $(t, p, q) \in \{1, \dots, 99\} \times \mathbb{R}^2$ . The result follows from Theorem 2.1.  $\square$

## References

- [1] Agarwal, Ravi P. On multipoint boundary value problems for discrete equations. *J. Math. Anal. Appl.* 96 (1983), no. 2, 520–534.
- [2] Agarwal, Ravi P. *Difference equations and inequalities. Theory, methods, and applications.* Second edition. Monographs and Textbooks in Pure and Applied Mathematics, 228. Marcel Dekker, Inc., New York, 2000.
- [3] Gaines, Robert. Difference equations associated with boundary value problems for second order nonlinear ordinary differential equations. *SIAM J. Numer. Anal.* 11 (1974), 411–434.
- [4] Henderson, J.; Thompson, H. B. Existence of multiple solutions for second-order discrete boundary value problems. *Comput. Math. Appl.* 43 (2002), no. 10-11, 1239–1248.
- [5] Henderson, Johnny; Thompson, H. B. Difference equations associated with fully nonlinear boundary value problems for second order ordinary differential equations. *J. Differ. Equations Appl.* 7 (2001), no. 2, 297–321.
- [6] Kelley, Walter G.; Peterson, Allan C. *Difference equations. An introduction with applications.* Second edition. Harcourt/Academic Press, San Diego, CA, 2001.
- [7] Lasota, A. A discrete boundary value problem. *Ann. Polon. Math.* 20 (1968), 183–190.
- [8] Lloyd, N. G. *Degree theory.* Cambridge Tracts in Mathematics, No. 73. Cambridge University Press, Cambridge-New York-Melbourne, 1978.

- [9] Mawhin, J.; Tisdell, C. C. A note on the uniqueness of solutions to nonlinear, discrete, vector boundary value problems. *Nonlinear analysis and applications: to V. Lakshmikantham on his 80th birthday*. Vol. 1, 2, 789–798, Kluwer Acad. Publ., Dordrecht, 2003.
- [10] Rachůnková, I.; Tisdell C.C. Existence of non-spurious solutions to discrete boundary value problems. *Aust. J. Math. Anal. Appl.* 3 (2006), no. 2, Art. 6, 1-9 pp. (electronic).
- [11] Rachůnková, I.; Tisdell C.C. Existence of non-spurious solutions to discrete Dirichlet problems with lower and upper solutions. *Nonlin. Anal.*, to appear. doi:10.1016/j.na.2006.07.010
- [12] Thompson, H. B. Topological methods for some boundary value problems. *Advances in difference equations, III*. *Comput. Math. Appl.* 42 (2001), no. 3-5, 487–495.
- [13] Thompson, H. B.; Tisdell, Christopher. Systems of difference equations associated with boundary value problems for second order systems of ordinary differential equations. *J. Math. Anal. Appl.* 248 (2000), no. 2, 333–347.
- [14] Thompson, H. B.; Tisdell, C. Boundary value problems for systems of difference equations associated with systems of second-order ordinary differential equations. *Appl. Math. Lett.* 15 (2002), no. 6, 761–766.
- [15] Thompson, H. B.; Tisdell, C. C. The nonexistence of spurious solutions to discrete, two-point boundary value problems. *Appl. Math. Lett.* 16 (2003), no. 1, 79–84.
- [16] Tisdell, C. C. The uniqueness of solutions to discrete, vector, two-point boundary value problems. *Appl. Math. Lett.* 16 (2003), no. 8, 1321–1328.