Time Scales[1]

We have previously considered how to get 'a priori' bounds on the solutions to various dynamic equations, without knowing anything on the existence of solutions. We now begin applying these bounds to existence questions.

Consider the following dynamic boundary value problem:

$$x^{\Delta} = f(t,x) \qquad t \in [a,c]_{\mathbb{T}} \tag{1}$$
$$M \cdot x(a) + R \cdot x(\sigma(c)) = \alpha \tag{2}$$

where $f : [a,c]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is continuous, $a, c \in \mathbb{T}$ and M, R, α are real-valued constants.

We want to find the conditions under which solutions exist to this system, and will look at the more general family:

$$x^{\Delta} = \lambda f(t,x) \qquad t \in [a,c]_{\mathbb{T}}$$
$$M \cdot x(a) + R \cdot x(\sigma(c)) = \lambda\alpha$$

where $\lambda \in [0,1]$. We denote this family by $\circledast$ (and for example $\circledast\big|_{\lambda=0}$ denotes the system of equations obtained from $\circledast$ by setting $\lambda = 0$).

This is useful because we now have the result:

Theorem 1. *Let f be continuous. If:*

(i) The family $\circledast$ has only the zero solution for $\lambda = 0$

(ii) For all $\lambda \in (0,1]$ all solutions to this family satisfy $|x(t)| < K \forall t \in [a, \sigma(c)]_{\mathbb{T}}$ for some $K > 0$ independent of λ

then there is at least one solution to the system $\circledast\big|_{\lambda=1}$ (i.e. our original system)[2].

Proof. Omitted (the theory involves topological ideas, e.g. degree theory and fixed point methods) □

Theorem 2. *Let f be continuous and $M + R \neq 0$. If there exists $B > 0$ such that both the following hold:*

- $\frac{1}{|R|}(|\alpha| + |M|B) \leq B$
- $xf(t,x) > 0$ *for all* $t \in [a,c]_{\mathbb{T}}$ *such that* $|x| \geq B$

then the dynamic Boundary Value Problem $\circledast\big|_{\lambda=1}$ has at least one solution.

Proof. Using the previous theorem, then:

- Given $\circledast\big|_{\lambda=0}$ then $x^{\Delta} = 0 \Rightarrow x(t) = A$ for some constant A, and then $M \cdot x(a) + R \cdot x(\sigma(c)) = 0 \Rightarrow (M+R)A = 0 \Rightarrow A = 0$ (since $M + R \neq 0$). Thus the only solution is the zero solution.

[1] These notes are from Tuesday 3^{rd} May, 2005, i.e. the 2nd lecture of Week 9. They are definitely not an exact transcription of what Chris put on the board, but do capture all the significant information. There are a couple of things I may have omitted or condensed as a matter of personal style.

[2] I've decided to go with this as the notation, since it works for me. I think the meaning still comes across, which is the important thing!

- We will assume we are seeking <u>continuous</u> solutions. Under this premise, define $r : [a\sigma(c)]_{\mathbb{T}} \to \mathbb{R}$ by $r(t) = x^2(t) - B^2$. We may choose $t_0 \in [a, \sigma(c)]_{\mathbb{T}}$ such that $(x(t_0))^2 \geq (x(t))^2$ for all $t \in [a, \sigma(c)]$ (existence of t_0 is guaranteed for a continuous function on a compact interval) and assume $(x(t_0))^2 \geq B^2$ (so that $|x(t_0)| \geq B$).

  Now if $t_0 \in [a, \sigma(c))_{\mathbb{T}}$ then $0 \geq (x^2(t))^\Delta\big|_{t=t_0}$ (since there is a maximum at t_0, this follows from the definition of the delta-derivative). But

  $$\begin{aligned} (x^2(t))^\Delta &= 2x(t)x^\Delta(t) + \mu(t)[x^\Delta(t)]^2 \quad \text{(this was demonstrated previously)} \\ &= 2x(t)\lambda f(t, x(t)) + \mu(t)(\lambda f(t, x(t)))^2 \\ &\geq 2\lambda x(t) f(t, x(t)) \end{aligned}$$

  and since $|x(t_0)| \geq B$ then

  $$(x^2(t))^\Delta\big|_{t=t_0} \geq 2\lambda x(t_0) f(t_0, x(t_0)) > 0, \quad \forall \lambda \in (0, 1]$$

  by our theorem's assumption.

  But now $0 \geq (x^2(t))^\Delta\big|_{t=t_0} > 0$, which gives a contradiction.

  If $t_0 = \sigma(c)$, on the other hand, then using the boundary condition $M \cdot x(a) + R \cdot x(\sigma(c)) = \lambda\alpha$ we have $x(\sigma(c)) = \frac{1}{R}(\lambda\alpha - M \cdot x(a))$, so

  $$|x(\sigma(c))| \leq \frac{1}{|R|}(|\alpha| + |M||x(a)|) < \frac{1}{|R|}(|\alpha| + |M|B) \leq B$$

  but by assumption $|x(\sigma(c))| = |x(t_0)| \geq B$, again giving a contradiction.

  Hence all possible continuous solutions to $\circledast\big|$ satisfy $|x(t)| < B$ for all $t \in [a, \sigma(c)]_{\mathbb{T}}$ and all $\lambda \in (0, 1]$.

This means all conditions of the previous theorem hold, with $K = B$, and so we conclude that the BVP $\circledast\big|_{\lambda=1}$ has at least one solution. □

Example 1. *Consider now the particular system:*

$$\begin{aligned} x^\Delta &= (t+1)(x^3+1) \qquad t \in [0, c]_{\mathbb{T}} \\ x(0) - 3x(\sigma(c)) &= 0 \end{aligned}$$

Find a possible value for B. (this is often the tricky part of any particular problem of this nature!)

See that $f(t, x) = (t+1)(x^3+1)$, $M = 1$, $R = -3$ and $\alpha = 0$. Can we find a $B > 0$ such that the conditions of our theorem are satisfied? Consider

$$xf(t, x) = (t+1)(x^4 + x).$$

Now $t + 1 > 0$ for all $t \in [0, c]_{\mathbb{T}}$ and $x^4 + x > 0$ for all $|x| = 2$. So if we make $B = 2$, does this mean that $\frac{1}{|R|}(|\alpha| + |M|B) \leq B$? Yes, this is easy to check! Hence all of the conditions of our existence theorem are satisfied and we conclude that our example has at least one solution.

Open Problem Can you remove the condition $\frac{1}{|R|}(|\alpha| + |M|B) \leq B$ from our existence theorem?