Math 5215 W7 MON

Last time: $e_p(t, t_0)$

Today: (More on) dynamic eqns on $\mathbb{T}$

Example: Consider $\mathbb{T} = \{n^2 \ni n \in \mathbb{N}_0\}$
$= \{0, 1, 2^2, \ldots\}$

We claim that $e_1(t, t_0) = (2^{\sqrt{t}})[(\sqrt{t})!]$

Ok, we need to show that $y(t) = (2^{\sqrt{t}})[(\sqrt{t})!]$

satisfies the dynamic IVP

① $y^\Delta = 1y$,
② $y(0) = 1$

$y(0) = (2^0 \cdot 0!) = 1$ so ② satisfies

Noted that all pts in $\mathbb{T}$ are r-s, so

$$y^\Delta(t) = \frac{y(\sigma(t)) - y(t)}{\sigma(t) - t}$$

Now find $\sigma(t)$

Let $t = n^2$ (so $n = \sqrt{t}$)

$$\begin{aligned}
\sigma(t) &= \inf\{s \in \mathbb{T} \ni s > t\} \\
\sigma(n^2) &= \inf\{s \in \mathbb{T} \ni s > n^2\} \\
&= \inf\{(n+1)^2, (n+2)^2, \ldots\} \\
&= (n+1)^2 \\
&= (\sqrt{t}+1)^2
\end{aligned}$$

$$y^{\Delta}(t) = \frac{y(\sigma(t)) - y(t)}{\sigma(t) - t}$$

$$= \frac{2^{\sqrt{\sigma(t)}}[(\sqrt{\sigma(t)})!] - 2^{\sqrt{t}}[(\sqrt{t})!]}{\sigma(t) - t}$$

$$= \frac{2^{\sqrt{t}+1}[(\sqrt{t}+1)!] - 2^{\sqrt{t}}[(\sqrt{t})!]}{(\sqrt{t}+1)^2 - t}$$

$$= \frac{2^{\sqrt{t}}[(\sqrt{t})!]\{-1+2(\sqrt{t}+1)\}}{t + 2\sqrt{t} + 1 - t}$$

$$= 2^{\sqrt{t}}[(\sqrt{t})!]$$
$$= y(t) \qquad \square$$

and therefore ① holds

Hence $e_1(t,0) = (2^{\sqrt{t}})[(\sqrt{t})!]$

since $y(t) = (2^{\sqrt{t}})[(\sqrt{t})!]$ solve ① and ②

We now consider a slightly wider class of dynamic IVPs with arbitrary initial condition, that is

③ $y^{\Delta} = p(t)\,y$

④ $y(t_0) = y_0$, $t_0 \in \mathbb{T}$, $y_0 \in \mathbb{R}$

Th$^{\underline{m}}$ 1 : Let $p \in \mathcal{R}$ (ie., $p \in C_{rd}$ and $1 + \mu p \neq 0$) Then the unique sol$^{\underline{n}}$ to the IVP ③, ④ is

⑤ $y(t) = e_p(t, t_0)\, y_0$

(This may be verfied by show ⑤ satisfies ③, ④).

Similarly, lets consider the dynamics IVP

$$⑥ \quad y^{\Delta} = -p(t)\, y^{\sigma} \qquad (y^{\sigma} = y \circ \sigma)$$
$$⑦ \quad y(t) = y_0 \qquad , t_0 \in \mathbb{T} , y_0 \in \mathbb{R}$$

Th^m 2: Let $p \in \mathcal{R}$. Then the unique soln to ⑥, ⑦ is

$$⑧ \quad y(t) = \frac{1}{e_p(t, t_0)} y_0$$

Proof: Show ⑧ satisfies ⑥, ⑦

$$Y(t_0) = \frac{1}{e_p(t_0, t_0)} y_0 = y_0$$

$$y^{\Delta}(t) = \left[\frac{y_0}{e_p(t, t_0)}\right]^{\Delta} = y_0 \left[\frac{1}{e_p(t, t_0)}\right]^{\Delta}$$

$$= y_0 \left[\frac{-p(t)}{e_p(\sigma(t), t_0)}\right] \quad \text{, from W6 TUE Th}^{m} 1 \ (v)$$

$$= -p(t) \left[\frac{y_0}{e^{\sigma}_p(t, t_0)}\right]$$

$$= -p(t)\, y^{\sigma}(t)$$

Now consider the more general dynamics IVP:

$$\text{(9)} \quad y^{\Delta} = -p(t)\, y^{\sigma} + f(t)$$
$$\text{(10)} \quad y(t_0) = y_0$$

Th$^{\underline{m}}$ 3: Let $p \in \mathcal{R}$ and $f \in C_{rd}$. The unique sol$^{\underline{n}}$ to (9), (10) is

$$\text{(11)} \quad y(t) = \frac{y_0 + \int_{t_0}^{t} e_p(s,t_0) f(s)\,\Delta s}{e_p(t,t_0)}$$

Proof: Let y be a sol$^{\underline{n}}$ to (9), (10) and consider

$$\begin{aligned}
[e_p(t,t_0)y(t)]^{\Delta} &= e_p(t,t_0)\,y^{\Delta}(t) + e_p^{\Delta}(t,t_0)\,y^{\sigma}(t) \\
&= e_p(t,t_0)\,y^{\Delta}(t) + p(t)\,e_p(t,t_0)\,y^{\sigma}(t) \\
&= e_p(t,t_0)\,[\underbrace{y^{\Delta}(t) + p(t)\,y^{\sigma}(t)}_{f}] \\
&= e_p(t,t_0)\,f(t) \qquad \text{, from (9)}
\end{aligned}$$

So taking delta-integrals of both sides from t_0 to t we get

$$\int_{t_0}^{t} [e_p(t,t_0)y(t)]^{\Delta}\,\Delta s = \int_{t_0}^{t} e_p(s,t_0)\,f(s)\,\Delta s$$

$$e_p(t,t_0)\,y(t) - e_p(t_0,t_0)\,y(t_0) = \int_{t_0}^{t} e_p(s,t_0) f(s)\,\Delta s$$

$$y(t) = \frac{y_0 + \int_{t_0}^{t} e_p(s,t_0) f(s)\,\Delta s}{e_p(t,t_0)}$$

Example Let $\mathbb{T}$ be arbitrary. Solve

(i) $y^\Delta = -y^\sigma + 1$, $y(t_0) = 2$, $t_0 \in \mathbb{T}$

See that this is in form ⑨, ⑩ with

$$p(t) = 1, \quad f(t) = 1, \quad y_0 = 2$$

$$y(t) = \frac{y(t_0) + \int_{t_0}^{t} e_p(s,t_0) f(s)\,\Delta s}{e_p(t,t_0)}$$

$$= \frac{2 + \int_{t_0}^{t} e_1(s,t_0)\,\Delta s}{e_1(t,t_0)}$$

$$= \frac{2 + [e_1(s,t_0)]_{t_0}^{t}}{e_1(t,t_0)}$$

// Recall: $e_1^\Delta(s,t_0) = \overset{1}{\overline{p(s)}}\, e_1(s,t_0) = e_1(s,t_0)$ //

$$= \frac{1 + e_1(t,t_0)}{e_1(t,t_0)}$$

(ii) $y^\Delta = -t y^\sigma + t$, $y(t_0) = 2$, $t_0 \in \mathbb{T}$

See this is in form ⑪ with

$$p(t) = t, \quad f(t) = t, \quad y_0 = 2$$

$$y(t) = \frac{y(t_0) + \int_{t_0}^{t} e_p(s,t_0) f(s)\,\Delta s}{e_p(t,t_0)}$$

$$= \frac{2 + \int_{t_0}^{t} e_t(s,t_0)\, s\,\Delta s}{e_p(t,t_0)}$$

since

$$= \frac{2 + [e_t(s,t_0)]_{s=t_0}^{s=t}}{e_p(t,t_0)} \to [e_t(s,t_0)]^\Delta = t e_t(t,t_0)$$

$$= \frac{1 + e_t(t,t_0)}{e_t(t,t_0)}$$

Open Question :

When can we simplify

$$\int_{t_0}^{t} e_p(s,t_0)\,\Delta s \quad ?$$

$p(t) = \text{const}$ is the "nicest" case.

Example Let $\mathbb{T}$ be arbitrary, Solve.

(iii) $$y^\Delta = -t y^\sigma + \frac{1}{e_t(t,t_0)}, \quad y(t_0) = 0$$

See that this is in form ⑨, ⑩ with $p(t) = t$ $f(t) = \frac{1}{e_t(t,t_0)}$, $y_0 = 0$

So ⑪ gives

$$y(t) = \frac{y(t_0) + \int_{t_0}^{t} e_p(s,t_0) f(s)\,\Delta s}{e_p(t,t_0)}$$

$$= \frac{\int_{t_0}^{t} e_t(s,t_0) \cdot \frac{1}{e_t(s,t_0)}\,\Delta s}{e_t(t,t_0)}$$

$$= \frac{t - t_0}{e_t(t,t_0)}$$