<u>Last time</u> : solved $y^\Delta = y$, $y(0)=1$ for a <u>general</u> Time scale.

<u>today</u> we want to solve $y^\Delta = p(t)y$, $y(t_0)=1$

consider the (more general) dynamic IVP given by

$$\text{(J)} \quad \begin{cases} y^\Delta = p(t)y & , t \in \mathbb{T}^K \\ y(t_0) = 1 \end{cases}$$

where $p : \mathbb{T} \to \mathbb{R}$, $t_0 \in \mathbb{T}$

If $\underline{p \in C_{rd}}$ and $\underline{1+\mu(t)p(t) \neq 0}$ $\forall t \in \mathbb{T}^K$

then the unique solution to (J) is defined by:

$$e_p(t, t_0)$$

which is called "the generalised exponential function on $\mathbb{T}$!"

We can also write $e_p(t,t_0)$ explicitly:

$$e_p(t,t_0) = \begin{cases} \exp\left[\int_{t_0}^{t} \frac{1}{\mu(\tau)} \operatorname{Log}[1+\mu(\tau)p(\tau)]\Delta\tau\right], & \mu > 0 \\ \exp\left[\int_{t_0}^{t} p(\tau)\, d\tau\right], & \mu = 0 \end{cases}$$

note here that Log is the <u>principle logarithm</u>

Note that in the previous lecture, we defined $e(t,0)$. We understand this to mean $e(t,0) = e_1(t,0)$

defn: Define $p: \mathbb{T} \to \mathbb{R}$ as regressive if $p \in C_{rd}$ and $1 + \mu(t) p(t) \neq 0$ $\forall t \in \mathbb{T}^{\kappa}$.

The space of all regressive functions is denoted $\mathcal{R}$ (funny-looking R)

Some useful properties of $e_p(t, t_0)$ are summarized in the following theorem:

THM 1 If $p, q \in \mathcal{R}$, then

(I) $e_0(t, t_0) \equiv 1$ and $e_p(t, t) = 1$

(II) $e_p(\sigma(t), t_0) = (1 + \mu(t) p(t)) e_p(t, t_0)$

(III) $e_p(t, t_0) = \dfrac{1}{e_p(t_0, t)}$

(IV) $e_p(t, t_0) e_p(t_0, r) = e_p(t, r)$

(V) $\left(\dfrac{1}{e_p(t, t_0)}\right)^{\Delta} = \dfrac{-p(t)}{e_p^{\sigma}(t, t_0)}$

where $e_p^{\sigma}(t, t_0) = e_p(\sigma(t), t_0)$

proof (II): we know $f^{\sigma} = f + \mu f^{\Delta}$

so,

$$e_p(\sigma(t), t_0) = e_p(t, t_0) + \mu(t) e_p^{\Delta}(t, t_0)$$

$$= e_p(t, t_0) + \mu(t) p(t) e_p(t, t_0)$$

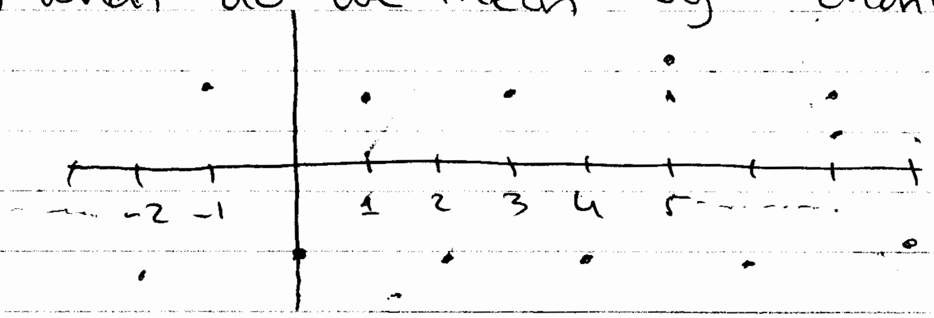
$$= (1 + p(t) \mu(t)) e_p(t, t_0)$$

example consider the IVP

$$(E)\quad \begin{cases} y^{\Delta} = ay, & t \in \mathbb{T}^{\kappa}, \quad \mathbb{T} \text{ arbitrary} \\ & a < -1 \\ y(0) = 1 & a = \text{constant} \end{cases}$$

for $\mathbb{T} = \mathbb{Z}$, show that the solution to (E) changes sign at every point in $\mathbb{Z}$.

* what do we mean by 'changes sign'?



well, a solution, y, to (E) $(\mathbb{T} = \mathbb{Z})$ changes sign when $y(t)\, y(\sigma(t)) < 0$.

Here, $p(t) \equiv a$, $t_0 = 0$

So our solution:

$$y(t) = e_p(t, t_0) = e_a(t, 0)$$

and

$$\begin{aligned} y(\sigma(t)) &= e_p(\sigma(t), t_0) \\ &= e_a(\sigma(t), 0) \\ &= (1 + \mu(t) a)\, e_a(t, 0) \end{aligned}$$

by THM 1 (11)

$$\begin{aligned} \text{So,}\quad y(t)\, y(\sigma(t)) &= e_a(t, 0)\, e_a(\sigma(t), 0) \\ &= e_a(t, 0)\, (1 + \mu(t) a)\, e_a(t, 0) \\ &= [e_a(t, 0)]^2 (1 + \mu(t) a) \\ &= [e_a(t, 0)]^2 (1 + a) \\ &\quad (\mu = 1, \ \mathbb{T} = \mathbb{Z}) \end{aligned}$$

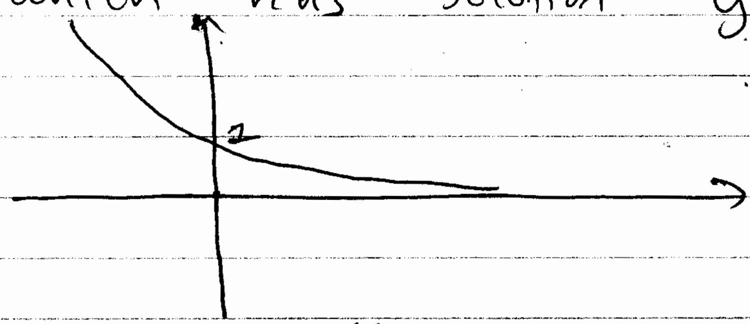
< 0 when $a < -1$

for general $\mathbb{T}$, we need $1 + \mu(t) a < 0$ for our solutions to "oscillate".

Lets compare our sol^n's to IVP using $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \mathbb{R}$. We already have $\mathbb{T} = \mathbb{Z}$. For $\mathbb{T} = \mathbb{R}$,

$$y' = ay, \quad a < -1$$
$$y(0) = 1$$

which has solution $y = e^{at}$



* so our "general" exponential in $\mathbb{Z}$ is oscillatory

examples of $e_p(t, t_0)$

(i) $\mathbb{T} = h\mathbb{Z}$, $h > 0$ $\quad p(t) = \alpha \in \mathbb{R}$ $= \text{constant}$.

then,

$$e_\alpha(t, 0) = (1 + \alpha h)^{\frac{t}{h}}$$

To verify this, we show $y(t) = e_\alpha(t, 0)$ satisfies:

$$y^\Delta = \frac{y(t+h) - y(t)}{h}$$

$$= \alpha\, y(t) = p(t)\, y(t)$$

and that $y(0) = 1$.

Consider initial condition first:

$$y(0) = e_\alpha(0,0) = (1+\alpha h)^0 = 1$$

$$y^\Delta(t) = \frac{y(t+h)-y(t)}{h} = \frac{(1+\alpha h)^{\frac{t+h}{h}} - (1+\alpha h)^{\frac{t}{h}}}{h}$$

$$= \frac{(1+\alpha h)^{\frac{t}{h}}(1+\alpha h) - (1+\alpha h)^{\frac{t}{h}}}{h}$$

$$= (1+\alpha h)^{\frac{t}{h}} \left(\frac{1+\alpha h-1}{h}\right)$$

$$= \alpha (1+\alpha h)^{\frac{t}{h}} = \alpha\, y(t)$$

as required.