MATH5215 CLASS NOTES FOR 05/4/2005 BY CHRISTIAN MOLDENHAUER

Last time we dealt with the Delta Integral. Today we will look at some examples and the Chain Rule on $\mathbb{T}$. Some special cases will be presented now.

Theorem: Let $a, b \in \mathbb{T}$ and $f \in C_{rd}$

(i) If $\mathbb{T} = \mathbb{R}$ then $\int_a^b f(t)\Delta t = \int_a^b f(t)\mathrm{d}t$

(ii) If $[a, b]_{\mathbb{T}}$ consists only of isolated points then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t\in[a,b)_{\mathbb{T}}} \mu(t)f(t) & \text{for } a < b \\ 0 & \text{for } a = b \\ -\sum_{t\in[b,a)_{\mathbb{T}}} \mu(t)f(t) & \text{for } a > b \end{cases}$$

(iii) If $\mathbb{T} = h\mathbb{Z} := \{hk : k \in \mathbb{Z}\}(h > 0)$ then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{k=a/h}^{b/h-1} f(kh)h & \text{for } a < b \\ 0 & \text{for } a = b \\ -\sum_{k=b/h}^{a/h-1} f(kh)h & \text{for } a > b \end{cases}$$

(iv) If $\mathbb{T} = \mathbb{Z}$ then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t=a}^{b-1} f(t) & \text{for } a < b \\ 0 & \text{for } a = b \\ -\sum_{t=b}^{a-1} f(t) & \text{for } a > b \end{cases}$$

Proof of (ii) (with $a < b$). Let $F^{\Delta} = f$. F exists because $f \in C_{rd}$. Furthermore we have $[a, b]_{\mathbb{T}} = \{a = t_0, t_1, \ldots, t_n = b\}$ where as $\sigma(t_0) = t_1$ etc. Hence

$$\begin{aligned} \int_a^b f(t)\Delta t &= \int_{a=t_0}^{t_1} f(t)\Delta t + \int_{t_1}^{t_2} f(t)\Delta t + \ldots + \int_{t_{n-1}}^{b=t_n} f(t)\Delta t \\ &= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(t)\Delta t = \sum_{i=0}^{n-1} \int_{t_i}^{\sigma(t_i)} f(t)\Delta t \\ &= \sum_{i=0}^{n-1} F(\sigma(t_i)) - F(t_i) = \sum_{i=0}^{n-1} \mu(t_i)F^{\Delta}(t_i) \\ &= \sum_{i=0}^{n-1} \mu(t_i)f(t_i) = \sum_{[a,b)_{\mathbb{T}}} \mu(t)f(t) \end{aligned}$$

Examples

(i) Let $a \in \mathbb{T}$, $\mathbb{T}$ arbitrary. Consider $\int_a^t 1\Delta s$. We know that $t^\Delta = 1$, so $F(t) = t$ is an anit-δ-derivative of 1. Thus

$$\int_a^t 1\Delta s = F(t) - F(a) = t - a$$

(ii) Evaluate $\int_0^t s\Delta s$ for $t, 0 \in \mathbb{T}$ and different Time Scales $\mathbb{T}$. We know by product rule for an arbitrary $\mathbb{T}$

$$\begin{aligned} (t^2)^\Delta &= t + \sigma(t), \qquad \text{therefore} \\ \int_a^t (s^2)^\Delta \Delta s &= \int_a^t s + \sigma(s)\Delta s, \qquad \text{and hence} \\ t^2 - a^2 &= \int_a^t s\Delta s + \int_a^t \sigma(s)\Delta s, \qquad \text{finally} \end{aligned}$$

$$\int_a^t s\Delta s = t^2 - a^2 - \int_a^t \sigma(s)\Delta s \tag{1}$$

With these results we can now consider serveral cases for different Time Scales.

(a) $\mathbb{T} = \mathbb{R}$, $t \in \mathbb{R}$, $\sigma(t) = t$. We get

$$(1) \Rightarrow \int_0^t s\Delta s = t^2 - \int_0^t s\mathrm{d}s = \frac{1}{2}t^2$$

(b) $\mathbb{T} = \mathbb{Z}$, $t \in \mathbb{Z}$, $\sigma(t) = t + 1$. We get

$$\begin{aligned} (1) \Rightarrow \int_0^t s\Delta s &= t^2 - \int_0^t (s+1)\Delta s \\ &= t^2 - \sum_{s=1}^{t-1} s + 1 = t^2 - \sum_{s=0}^{t-1} s - \sum_{s=0}^{t-1} 1 \\ &= t^2 - \left[\frac{s^{\underline{2}}}{2}\right]_{s=0}^{s=t} - \Big[s\Big]_{s=0}^{s=t} \\ &= \frac{t^2}{2} - \frac{1}{2}t \end{aligned}$$

Hereby we used the relationships: $s = s^{\underline{1}}$ and $\sum s^{\underline{1}} = \frac{s^{\underline{2}}}{2}$.

(c) $\mathbb{T} = h\mathbb{Z}$, $h > 0$, $t \in h\mathbb{Z}$, $\sigma(t) = t + h$ and $\mu(t) = h$. We get

$$
\begin{aligned}
(1) \Rightarrow \int_0^t s\Delta s &= t^2 - \int_0^t \sigma(s)\Delta s = t^2 - \int_0^t (s+h)\Delta s \\
&= t^2 - \sum_{s=0}^{t/h-1} h(sh+h) \\
&= t^2 - h^2 \left[\sum_{s=0}^{t/h-1} s + \sum_{s=0}^{t/h-1} 1 \right] \\
&= t^2 - h^2 \left[\left[\frac{s^{\underline{2}}}{2} \right]_0^{\frac{t}{h}} + \left[s \right]_0^{\frac{t}{h}} \right] \\
&= t^2 - h^2 \left[\frac{\frac{t}{h}\left(\frac{t}{h} - 1\right)}{2h} + \frac{t}{h} \right]
\end{aligned}
$$

(d) $\mathbb{T} = [0,1] \cup [2,3]$, $t \in \mathbb{T}$ and $\sigma(t) = \begin{cases} t \text{ for } t \in [0,1) \cup [2,3] \\ t+1 \text{ for } t = 1 \end{cases}$

- First case: $t \in [0,1]$, $\sigma(t) = t$, $\sigma(1) = 1$

$$
\begin{aligned}
(1) \Rightarrow \int_0^t s\Delta s &= t^2 - \int_0^t \sigma(s)\Delta s \\
&= t^2 - \int_0^t s\mathrm{d}s = \frac{1}{2}t^2
\end{aligned}
$$

- Second case: $t \in [2,3]$. We have to obtain a variant of (1) by using $a = 2$.

$$
\begin{aligned}
\int_2^t (s^2)^\Delta \Delta s &= \int_2^t s + \sigma(s)\Delta s, \text{gives} \\
\int_2^t s\Delta s &= t^2 - 4 - \int_2^t s\mathrm{d}s \\
&= \frac{1}{2}t^2 - 2
\end{aligned}
$$

So we have in this case

$$
\begin{aligned}
\int_0^t s\Delta s &= \int_0^2 s\Delta s + \int_2^t s\Delta s \\
&= \int_0^1 s\Delta s + \sum_{s=1}^{1} s^{\underline{1}} + \int_2^t s\Delta s \\
&= \frac{1}{2} + \left[\frac{s(s-1)}{2} \right]_1^2 + \frac{1}{2}t^2 - 2 \\
&= \frac{1}{2}t^2 - \frac{1}{2}
\end{aligned}
$$