MATH5215 Notes: Week 5 Monday 04/04/05 by Smith Huang

Last Time: Regulated and C_{rd} function.
Today: More on integration on $\mathbb{T}$.

Definition: A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called pre-delta-diff'able on a set D (with region of d'bility D) if:

a) $D \subset \mathbb{T}^\kappa$
b) $\mathbb{T}^\kappa \backslash D$ is countable and contains no r-s elements of $\mathbb{T}$
c) f is d-diff'able on D

Example: If $\mathbb{T}$ have only isolated points then $D = \mathbb{T}^\kappa$.

Exercise: If $\mathbb{T} = \mathbb{R}$

$$f = \begin{cases} 0 & t = 0 \\ \frac{1}{t} & t \in \mathbb{R}\backslash\{0\} \end{cases}$$

then find D.

Theorem 1:

If $f : [a,b]_\mathbb{T} \to \mathbb{R}$ (Notes: $[a,b]_\mathbb{T} = [a,b] \cap \mathbb{T}$) is a regulated on then closed and bounded (compact) intervals $[a,b]_\mathbb{T}$ then f is bounded on $[a,b]_\mathbb{T}$.
(i.e. $\exists$ R ≥ 0 s.t. $\sup_{t\in[a,b]_\mathbb{T}} |f(t)| \leq$ R)

Theorem 2: (MVT for $\mathbb{T}$)

Let $f, g : \mathbb{T} \to \mathbb{R}$ with both functions pre-d-diff'able on D, if

$$|f^\Delta(t)| \leq g^\Delta(t), \quad \forall t \in D$$

then

$$|f(s) - f(r)| \leq g(s) - g(r), \quad \forall r, s \in \mathbb{T}, \quad r \leq s$$

Corollary 1: Let f, g, be pre-d-diff'able with region D.

1. If U is a compact interval with end points r,s $\in \mathbb{T}$ then

$$|f(s) - f(r)| \leq \left\{ \sup_{t\in U^\kappa \cap D} |f^\Delta(t)| \right\} |s - r|$$

2. If $f^\Delta(t) = 0$ $\forall t \in D$ then $f \equiv$ constant

3. If $f^\Delta(t) = g^\Delta(t)$ $\forall t \in D$ then $f(t) = g(t) + c$ $\forall t \in D$ where $c =$ constant

Proof:

1. Define g

$$g(t) = \left\{ \sup_{\tau \in [r,s]_{\mathbb{T}}^{\kappa} \cap D} |f^{\Delta}(\tau)| \right\} (t - r), \quad \forall t \in \mathbb{T}$$

   then

$$|f^{\Delta}(t)| \leq \sup_{\tau \in [r,s]_{\mathbb{T}}^{\kappa} \cap D} |f^{\Delta}(\tau)| = g^{\Delta}(t), \quad \forall t \in D \cap [r,s]_{\mathbb{T}}^{\kappa}$$

   So by Theorem 2:

$$g(t) - g(r) \geq |f(t) - f(r)|, \quad \forall t \in [r,s]_{\mathbb{T}}^{\kappa}$$

$$\begin{aligned} |f(s) - f(r)| &\leq g(s) - g(r) = g(s) \\ &= \left\{ \sup_{\tau \in [r,s]_{\mathbb{T}}^{\kappa} \cap D} |f^{\Delta}(\tau)| \right\} (s - r) \end{aligned}$$

2. follows from 1.

3. follows from 2.

Theorem 3: (Existence of pre-anti-d-deriv's)
If $f : \mathbb{T} \to \mathbb{R}$ is a regulated function then $\exists$ a function F, which is pre-d-diff'able with region D s.t.

$$F^{\Delta}(t) = f(t), \quad \forall t \in D$$

Definition: Let $f : \mathbb{T} \to \mathbb{R}$ be regulated. Any function F as in Theorem 3 is called a pre-anti-delta-derivative of f. We define the indefinite delta integral of f by:

$$\int f(t)\,\Delta t = F(t) + c, \quad c = constant$$

and F is a pre-anti-delta derivative of f.
We define the Cauchy delta integral by

$$\int_s^r f(t)\,\delta t = F(r) - F(s) \quad \forall r, s \in \mathbb{T}$$

Finally, a function is called anti-delta-derivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\Delta}(t) = f(t)\ \forall t \in \mathbb{T}^{\kappa}$

Theorem 4: (Existence of anti-d-dervative)

If $f \in C_{rd}$ then f has an anti-d-derivative. In particular, if $t_0 \in \mathbb{T}$ then

$$F(t) = \int_{t_0}^{t} f(\tau)\,\Delta\tau, \quad \forall t \in \mathbb{T}$$

is an anti-d-derivative of f.

Proof: (Use definition of d-diff'ability)
Let $f \in C_{rd}$. Then f is also regulated. Let F be the function guaranteed to exist by Theorem 3 such that,

$$F^{\Delta}(t) = f(t) \quad \forall t \in D$$

We show $F^{\Delta}(t) = f(t)\ \forall t \in \mathbb{T}^{\kappa}$.
Obviously, $F^{\Delta}(t) = f(t)$, $\forall t \in D$ from above so consider $t \in \mathbb{T}^{\kappa} \backslash D$. The point must be r-d because $\mathbb{T}^{\kappa} \backslash D$ cannot contain any r-s points. Since t is r-d and $f \in C_{rd}$, we must have f continuous at t.
Let $\epsilon > 0$, Then $\exists$ nbd U of t with

$$|f(s) - f(t)| \leq \epsilon \quad \forall s \in U$$

Let $h(\tau) = F(\tau) - f(t)(\tau - t_0)\ \forall \tau \in \mathbb{T}$
The h is pre-d-diff'able with region D and

$$h^{\Delta}(\tau) = F^{\Delta}(\tau) - f(t) = f(\tau) - f(t), \quad \forall \tau \in D$$

Hence

$$|h^{\Delta}(s)| \leq |f(s) - f(t)| \leq \epsilon, \quad \forall s \in D \cap U$$

So

$$\sup |h^{\Delta}(s)| \leq \epsilon, \quad \forall s \in D \cap U$$

Hence by Corollary 1, $\forall r \in U$

$$\begin{aligned} |F(t) - F(r) - f(t)(t-r)| &= |h(t) + f(t)(t-t_0) - [h(r) + f(t)(r-t_0)] - f(t)(t-r)| \\ &= |h(t) - h(r)| \\ &\leq \left\{ \sup_{S \in D \cap U} |h^{\Delta}(s)| \right\} |t-r| \\ &\leq \epsilon |t-r| \end{aligned}$$

So F is d-diff'able with $F^{\Delta}(t) = f(t)$

Theorem 5:

If $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, $f, g \in C_{rd}$.

1. $\int_a^b (f(t) + g(t))\,\Delta t = \int_a^b f(t)\,\Delta t + \int_a^b g(t)\,\Delta t$
2. $\int_a^b \alpha f(t)\,\Delta t = \alpha \int_a^b f(t)\,\Delta t$
3. $\int_a^b f(t)\,\Delta t = -\int_b^a f(t)\,\Delta t$
4. $\int_a^b f(t)\,\Delta t = \int_a^c f(t)\,\Delta t + \int_c^b f(t)\,\Delta t$
5. $\int_a^b f(\sigma(t))g^{\Delta}(t)\,\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^{\Delta}(t)g(t)\,\Delta t$
6. $\int_a^b f(t)g^{\Delta}(t)\,\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\,\Delta t$
7. $\int_a^a f(t)\,\Delta t = 0$