Revision

problem sheet 5

① $y^{\Delta} = p(t)\,y$, $t \in \mathbb{T}^{k}$, $p \in \mathcal{R}$

$y(t_0) = 1$, $t_0 \in \mathbb{T}$

Let y be a solution to the above and show:

$$\left[\frac{y(t)}{e_p(t,t_0)}\right]^{\Delta} = 0$$

so $$\left[\frac{y(t)}{e_p(t,t_0)}\right]^{\Delta} = \frac{y^{\Delta}(t)e_p(t,t_0) - y(t)\,e_p^{\Delta}(t,t_0)}{e_p(t,t_0)\,e_p(\sigma(t),t_0)}$$

(by quotient rule)

$$= \frac{\cancel{p(t)\,y(t)\,e_p(t,t_0) - y(t)p(t)e_p(t,t_0)}}{e_p(t,t_0)\,e_p(\sigma(t),t_0)}$$

$$= 0$$

now, $$\left[\frac{y(t)}{e_p(t,t_0)}\right]^{\Delta} = 0$$

$$\Rightarrow \quad \frac{y(t)}{e_p(t,t_0)} = K = \text{const.}$$

so, we want to show $K = 1$

but $y(t_0) = 1$ and $e_p(t_0,t_0) = 1$

so $$\frac{y(t_0)}{e_p(t_0,t_0)} = 1 \Rightarrow K = 1$$

② $\mathbb{T} = q^{\mathbb{N}_0} = \{1, q, q^2, \cdots\}$, $q>1$ const

what is $e_\alpha(t,1)$ for $\mathbb{T} = 2^{\mathbb{N}_0}$?

well, we know that $e_\alpha(t,t_0)$ solves

$y^\Delta = \alpha y$, $t \in \mathbb{T}^\kappa$

$y(t_0) = 1$ $t_0 \in \mathbb{T}$

all points in $\mathbb{T}$ are isolated

$\therefore$ $y^\Delta(t) = \dfrac{y(\sigma(t)) - y(t)}{\sigma(t) - t}$

for our $\mathbb{T}$, $\sigma(t) = qt$

$\mu(t) = (q-1)t$

rearrange $y^\Delta = \alpha y$ to get

$$y(\sigma(t)) = y(t) + \mu(t)\alpha y(t)$$

i.e. $y(qt) = [1 + \alpha(q-1)t]\, y(t)$

solve this recursively, and get

$$y(t) = \prod_{s = t_0, q t_0, q^2 t_0, \dots, \frac{t}{q}} [1 + (q-1)\alpha s]$$

$$= e_p(t, t_0)$$

for $\mathbb{T} = 2^{\mathbb{N}_0}$, take $q=2$ above.

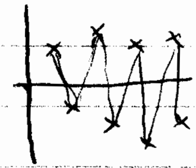
③ $y^\Delta = p(t)y \quad , p \in \mathcal{R}$

$y(t_0) = y_0, \quad t_0 \in \mathbb{T} \quad , \quad y_0 \in \mathbb{R}$

To make proper use of our solutions, we will assume that $\mathbb{T}$ contains only isolated points (this assumption turns out to be useful).

(i) oscillations: for a solution to oscillate <u>at every point</u> in $\mathbb{T}^K$, we must have

$$y(t)\, y(\sigma(t)) < 0$$



we know our solutions are $y(t) = e_p(t, t_0)$

so $\quad y(t)\, y(\sigma(t)) < 0$

$\quad \Rightarrow e_p(t,t_0)\, e_p(\sigma(t), t_0) < 0$

we know $\quad e_p^\Delta = e_p\, p$

$$\frac{e_p^\sigma - e_p}{\mu(t)} = e_p^\Delta$$

$$\Rightarrow e_p^\sigma = \mu(t)p(t)\, e_p + e_p$$

so $\quad e_p(t,t_0)\,(1+\mu(t)p(t))\, e_p(t,t_0) < 0$

i.e. $\quad [e_p(t,t_0)]^2\,(1+\mu(t)p(t)) < 0$

i.e. $\quad 1+\mu(t)p(t) < 0 \quad \forall t \in \mathbb{T}^K \quad$ for oscillations at each point.

for solutions to oscillate at least once in $\mathbb{T}^k$ we must have: everything the same in our calculations, except we say

$$\exists\ t \in \mathbb{T}^k \quad \text{s.t.} \quad 1+\mu(t)p(t) < 0$$

i.e. condition not there "$\forall\, t \in \mathbb{T}^k$".

iii) Non-oscillations
for a solution $y(t)$ never to oscillate we must have

$$y(t)\ y(\sigma(t)) > 0 \quad \forall\, t \in \mathbb{T}^k$$

i.e. $e_p(t,t_0)\, e_p(\sigma(t),t_0) > 0$

so $[e_p(t,t_0)]^2\ (1+\mu(t)p(t)) > 0$

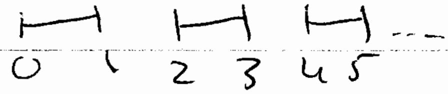
so $1+\mu(t)p(t) > 0 \quad \forall\, t \in \mathbb{T}^k$

i.e. same as strictly oscillating, but signs in opposite direction.

iii) for positive solutions, there is not much more work to be done. Because we know our solution is the exponential function, and $1+\mu(t)p(t) > 0$. If we start at $y_0 > 0$, and since we have a non-oscillatory case, then our solutions will ALWAYS be positive. (strictly positive)

iv) for negative solutions, same as above, except start at $y_0 < 0$, then solutions will stay negative

④ $\mathbb{T} = \bigcup_{k=0}^{\infty} [2k, 2k+1]$



the model obeys

$$N' = N \quad \forall t \in \bigcup_{k=0}^{\infty} [2k, 2k+1)$$

$$N(t+1) = 2N(t) \quad \text{for } t = 1, 3, 5, \ldots$$

$$\Rightarrow \quad N(t+1) - N(t) = N(t)$$

$$\Rightarrow \quad \Delta N(t) = N(t)$$

So combining these together gives:

$$N^{\Delta} = N \qquad t \in \mathbb{T}$$

so using IC $N(0) = 1$, we have solution:

$$N(t) = e(t, t_0)$$

⑤ easy - solve recursively

⑥ basically asked to verify first order linear Dynamic equations system.

Consider $\quad y^{\Delta} = -p(t)\, y^{\sigma} + f(t)$

$$y(t_0) = y_0$$

so let y be a solution, and consider

$$[y(t)\, e_p(t, t_0)]^{\Delta}$$

$$= y^{\Delta}(t)\, e_p(t, t_0) + y^{\sigma}(t)\, p(t)\, e_p(t, t_0)$$

$$= e_p(t,t_0)\left[y^\Delta(t) + p(t)\, y^\sigma(t)\right]$$

$$= e_p(t,t_0)\, f(t)$$

so Δ-integrate $e_p(t,t_0) f(t)$ to get

$$y(t)\, e_p(t,t_0) - y(t_0)\, \overbrace{e_p(t_0,t_0)}^{=1}$$

$$= \int_{t_0}^{t} e_p(s,t_0)\, f(s)\, \Delta s$$

so $$y(t) = \frac{y(t_0) + \int_{t_0}^{t} e_p(s,t_0)\, f(s)\, \Delta s}{e_p(t,t_0)}$$