MATH5215 CLASS NOTES FOR 17/5/2005 BY CHRISTIAN MOLDENHAUER

Consider the dynamic IVP with arbitrary $\mathbb{T}$.

$$x^\Delta = f(t,x) \qquad t \in [t_0, N]_\mathbb{T} \tag{1}$$
$$x(t_0) = x_0 \qquad x_0 \in \mathbb{R}^n,\ t_0 \in \mathbb{R},\ N \in \mathbb{T} \tag{2}$$

where $f : [t, N]_\mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is continous and solutions to (1) and (2) are of type $x : [t_0, \sigma(N)]_\mathbb{T} \to \mathbb{R}^n$ and continous.

Question: If solutions to (1) and (2) exist, when are they unique?

Theorem 1: Let $\alpha > 0$ be a constant such that

$$\|f(t,x_2) - f(t,x_1)\| \le \alpha\|x_2 - x_1\| \qquad \forall t \in [t_0, N]_\mathbb{T}\,,\ \forall x_1, x_2 \in \mathbb{R}^n \text{ and} \tag{3}$$
$$\alpha\,[\sigma(N) - t_0] < 1 \tag{4}$$

then the IVP (1) and (2) has at most one solution.

Proof: Let $x_!$ and x_2 be two solutions to (1) and (2) and define $u(t) := x_2(t) - x_1(t)$. We write (1) and (2) in the more convenient form

$$x(t) - x(t_0) = \int_{t_0}^{t} x^\Delta(s)\Delta s, \qquad t \in [t_0, \sigma(N)]_\mathbb{T}$$

So $x(t) = \int_{t_0}^t f(s, x(s))\Delta s + x_0$ and we must have the following

$$u(t) = \int_{t_0}^{t} f(s, x_2(s)) - f(s, x_1(s))\Delta s$$

$$\begin{aligned}
\text{and hence}\quad \|u(t)\| = \|x_2(t) - x_1(t)\| &\le \int_{t_0}^{t} \|f(s, x_2(s)) - f(s, x_1(s))\|\Delta s \\
&\le \int_{t_0}^{\sigma(N)} \alpha\|x_2(t) - x_1(t)\|\Delta s \quad \text{by Lipschitz condition (3)} \\
&\le \alpha \sup_{t\in[t_0,\sigma(N)]} \|x_2(t) - x_1(t)\|\,(\sigma(N) - t_0)
\end{aligned}$$

So we get

$$\begin{aligned}
\sup_{t\in[t_0,\sigma(N)]} \|x_2(t) - x_1(t)\| &\le \alpha \sup_{t\in[t_0,\sigma(N)]} \|x_2(t) - x_1(t)\|\,(\sigma(N) - t_0) \\
0 &\le \sup_{t\in[t_0,\sigma(N)]} \|x_2(t) - x_1(t)\|\,[\alpha\,(\sigma(N) - t_0) - 1]
\end{aligned}$$

With (4) it follows that $\sup_{t\in[t_0,\sigma(N)]} \|x_2(t) - x_1(t)\| = 0$ and thus $x_1(t) = x_2(t)\,\forall t \in [t_0, \sigma(N)]_\mathbb{T}$. So if the IVP has solutions, they are unique.

Example: Consider

$$\begin{aligned}
x^\Delta &= tx \qquad \forall t \in [0, N]_\mathbb{T},\ 0, N \in \mathbb{T} \\
x(0) &= x_0 \qquad x \in \mathbb{R}
\end{aligned}$$

Can you find an α such that the Lipschitz condition (3) holds? ($\alpha = N$ ☺)

Theorem 2: If f satisfies

$$2\langle x_2 - x_1, f(t,x_2) - f(t,x_1)\rangle + \mu(t)\|f(t,x_2) - f(t,x_1)\|^2 \le 0,\ \forall t \in [t_0, N]_\mathbb{T},\ \forall x_1, x_2 \in \mathbb{R}^n \tag{5}$$

then the IVP (1) and (2) has at most one solution on $t \in [t_0, N]_\mathbb{T}$.

Proof: Let x_1 and x_2 be two solutions to (1) and (2) and $r(t) := \|x_2(t) - x_1(t)\|^2$, $\forall t \in [t_0, \sigma(N)]_{\mathbb{T}}$. From (2) we see, that $r(t_0) = 0$. Now show $r(t)^\Delta \leq 0$. Consider

$$\begin{aligned} r(t)^\Delta &= \langle x_2(t) - x_1(t), x_2(t) - x_1(t)\rangle^2 \\ &= 2\langle x_2(t) - x_1(t), f(t, x_1) - f(t, x_2) + \mu(t)\|f(t, x_1)f(t, x_2)\|^2 \\ &\leq 0, \qquad \text{by assumption (use product rule and } x^\sigma = x - \mu x^\Delta). \end{aligned}$$

So we have $r(t_0) = 0$, $r(t) \geq 0$ and $r^\Delta \leq 0$ $\forall t \in [t, N]_{\mathbb{T}}$. Thus $r = 0$ holds for all $t \in [t, N]_{\mathbb{T}}$. Hence $x_1 = x_2$, $t \in [t, \sigma(N)]_{\mathbb{T}}$.

Remark: In fact you can easily obtain uniqueness of solutions on $[t_0, \infty)_{\mathbb{T}}$.

Theorem 3: If f satisfies

$$2\langle x_2 - x_1, f(t, x_2) - f(t, x_1)\rangle + \mu(t)\|f(t, x_2) - f(t, x_1)\|^2 \leq 0,\ \forall t \in [t_0, \infty)_{\mathbb{T}},\ \forall x_1, x_2 \in \mathbb{R}^n \tag{6}$$

then the IVP (1) and (2) has at most one solution on $t \in [t_0, \infty)_{\mathbb{T}}$.

Theorem 4: If there exists a delta differentiable function $V : \mathbb{R}^n \to [0, \infty)$ such that

$$V(x) = 0 \text{ iff } x = 0 \tag{7}$$

$$[V(x)]^\Delta \leq 0 \text{ for all } (t, x) \in [t, \infty)_{\mathbb{T}} \times \mathbb{R}^n \tag{8}$$

then there is at most one solution to (1) and (2) on $t \in [t_0, \infty)_{\mathbb{T}}$.

Remark: See that $V(x) = \|x\|^2$ in Theorem 2 and 3 has all these qualities.
Notice we have tried to get uniqueness on $[t_0, \infty)_{\mathbb{T}}$ or $[t_0, \sigma(N)]_{\mathbb{T}}$

Question: Can you extend Theorems 1-4 to cover the case, when we want to gain uniqueness of solutions on $[\rho(K), t_0]_{\mathbb{T}}$ or $(-\infty, t_0]_{\mathbb{T}}$?