MATH5215 Notes: Week 11 Monday 16/05/05

Last Time: Consider the 2nd-order dynamic equation

$$(1) \quad y^{\Delta\Delta} + by^{\Delta} + cy = 0, \qquad \text{b, c = constants}$$

when $b^2 - 4c > 0$ (and $c\mu - b \in \Re$) we looked at example of (1)
Today: We partly answer what if $b^2 - 4c < 0$?

Consider the simpler case

$$(2) \quad y^{\Delta\Delta} + \gamma^2 y = 0, \qquad \gamma > 0 \text{ constant}$$

The characteristic equation is

$$\lambda^2 + \gamma^2 = 0$$

$\lambda = \pm\gamma i$, $i^2 = -1$ complex roots.
For this case, the general solution to (1) is

$$\begin{aligned}(3) \qquad y(t) &= Ae_{\lambda_1}(t, t_0) + Be_{\lambda_2}(t, t_0) \\ &= Ae_{i\gamma}(t, t_0) + Be_{-i\gamma}(t, t_0)\end{aligned}$$

The appearance of i and $-i$ in the general exponential function makes no significant difference to our knowledge of $e_p(t, t_0)$

$$\begin{aligned}\text{e.g.} \qquad [e_{i\gamma}(t, t_0)]^{\Delta} &= i\gamma e_{i\gamma}(t, t_0) \\ [e_{-i\gamma}(t, t_0)]^{\Delta} &= -i\gamma e_{-i\gamma}(t, t_0) \\ [e_{i\gamma}(t, t_0)]^{\Delta\Delta} &= -\gamma^2 e_{i\gamma}(t, t_0) \\ [e_{-i\gamma}(t, t_0)]^{\Delta\Delta} &= -\gamma^2 e_{-i\gamma}(t, t_0)\end{aligned}$$

so (3) is a solution to (2)

Q: How does a solution like (3) compare with a solution to (2) in $\mathbb{T} = \mathbb{R}$?
If $\mathbb{T} = \mathbb{R}$ then (2) becomes

$$y'' + \gamma^2 y = 0$$

$$\begin{aligned}\lambda &= \pm i\gamma \\ &= \alpha \pm i\beta\end{aligned}$$

we know the general solution is

$$\begin{aligned}(4) \qquad y(t) &= e^{\alpha t}(C\cos\beta t + D\sin\beta t) \\ &= Ccos\gamma t + Dsin\gamma t\end{aligned}$$

Q: Can we connect (3) and (4) in some natural way?
A: Yes, by introducing a general *sin* and *cos* function

Definition: If $p \in C_{rd}$ and $\mu p^2 \in \Re$ then define cos_p and sin_p by:

$$
\begin{aligned}
(5) \qquad cos_p &= \frac{e_{ip} + e_{-}ip}{2} \\
sin_p &= \frac{e_{ip} - e_{-}ip}{2i}
\end{aligned}
$$

Lemma: let $p \in C_{rd}$, if $\mu p^2 \in \Re$ then

$$
\begin{aligned}
\cos_p^{\Delta} &= -psin_p \\
sin_p^{\Delta} &= pcos_p
\end{aligned}
$$

Proof:

$$
\begin{aligned}
\cos_p^{\Delta} &= \left[\frac{e_{ip} + e_{-ip}}{2}\right]^{\Delta} \\
&= p\left[\frac{i}{2}(e_{ip} + e_{-ip})\right] \\
&= p\left[-\frac{1}{2i}(e_{ip} + e_{-ip})\right] \\
&= -psin_p
\end{aligned}
$$

Remark: From (5) it is easy to prove a general *Euler's formula*

$$e_{ip}(t.t_0) = cos_p(t, t_0) + isin_p(t, t_0)$$

Theorem: If $t_0 \in \mathbb{T}^{\kappa}$ then

$$(6) \quad y(t) = Ccos_{\gamma}(t, t_0) + Dsin_{\gamma}(t, t_0)$$

is a general solution to (2).
Note: Compare (6) with (4)

Example:

1. $\mathbb{T} = \mathbb{Z}$, for $p = \alpha$ a constant
   $sin_{\alpha}(t, 0) =?$, $\cos_{\alpha}(t, 0) =?$. Find $e_{\alpha}(t, 0)$?

   Solve $y^{\Delta} = \alpha y, \qquad y(0) = 1$
   For $\mathbb{T} \in \mathbb{Z}$

   $$\Delta y(t) = \alpha y(t), \qquad y(0) = 1$$

Solve recursively

$$\begin{aligned} y(t) &= (1+\alpha)^t \\ &= e_\alpha(t,0) \end{aligned}$$

From (5)

$$\begin{aligned} \sin_\alpha(t,0) &= \frac{e_{i\alpha}(t,0) - e_{-i\alpha}(t,0)}{2i} \\ &= \frac{(1+i\alpha)^t - (1-i\alpha)^t}{2i} \\ \cos_\alpha(t,0) &= \frac{e_{i\alpha}(t,0) + e_{-i\alpha}(t,0)}{2} \\ &= \frac{(1+i\alpha)^t + (1-i\alpha)^t}{2} \end{aligned}$$

2. If $\mathbb{T} = \mathbb{R}$ then

$$\begin{aligned} \sin_\alpha(t,0) &= \sin(\alpha t) \\ \cos_\alpha(t,0) &= \cos(\alpha t) \end{aligned}$$

3. *(Exercise)*
   Find $e_1(t,0), \sin_1(t,0)$ and $\cos_1(t,0)$ for $\mathbb{T} = \mathbb{Z}^2 = \{k^2 : k \in \mathbb{Z}\}$

Q: It seems natural to ask if $\cos_p$ and $\sin_p$ satisfy any natural identities? From Example 1, see that $\sin_p^2 + \cos_p^2$ may not equal 1.

Q: What if $b^2 - 4ac = 0$ in (1)? (This will mean roots of characteristics equation are equal).
If $\mathbb{T} = \mathbb{R}$ then we try general solutions of type

$$y(t) = Ae^{\lambda t} + Bte^{\lambda t}$$

For general time scale case we try solutions of form

$$y(t) = v(t)e_\lambda(t, t_0)$$

for a suitable function v.