DYNAMIC EQUATIONS OF TIME SCALES - LECTURE 1, WEEK 10

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May 9, 2005

So far our interest has been on dynamic equations of 'first order'. Namely,$y^{\triangle} = f(t, y, y^{\sigma})$ for f linear or non-linear.

We now consider (linear) second-order dynamic equations of the type

$$y^{\triangle\triangle} + p(t)y^{\triangle} + q(t)y = f(t) \tag{1}$$

where $p, q, f \in C_{rd}$ and $t \in \mathbb{T}^{\kappa^2}$.

If $f(t) = 0$ for all $t \in \mathbb{T}^{\kappa^2}$, then (1) is homogenous. Then

$$y^{\triangle\triangle} + p(t)y^{\triangle} + q(t)y = 0 \tag{2}$$

Otherwise we say that (1) is inhomogenous. A solution to (1) is a function $y : \mathbb{T} \to \mathbb{R}$ such that (1) holds and such that $y^{\triangle\triangle}$ exists with $y \in C_{rd}$. Denote this space by $y \in C^2_{rd}$.

Theorem 1. *[Superposition Principle]*

1. *If y_1 and y_2 solve (2), then $y = \alpha y_1 + \beta y_2$ is also a solution to (2) with α, β constant.*
2. *If y_3 is a solution to (2), and y_4 is a solution to (1) then $y = y_3 + y_4$ is a solution to (1).*

Definition 1. *[Regressiveness]*
We call (1) regressive, if $p, q \in C_{rd}$ satisfy $1 - \mu(t)p(t) + \mu^2(t)q(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa}$.

Theorem 2. *If (1) is regressive then the IVP*

$$y^{\triangle\triangle} + p(t)y^{\triangle} + q(t)y = f(t) \tag{3}$$

$$y(t_0) = y_0 \quad y^{\triangle}(t_0) = y_0^{\triangle} \tag{4}$$

has a unique solution y. Here y_0 and $y_0^{\triangle}$ are given constants.

Let us try to solve the homogenous IVP (2) with (4). Let y_1 and y_2 be two solutions to (2). Then by Theorem 1, $y(t) = \alpha y_1(t) + \beta y_2(t)$ is a solution for all $\alpha, \beta \in \mathbb{R}$. It turns out that α, β are solutions to the following linear system:

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1^{\triangle}(t_0) & y_2^{\triangle}(t_0) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0^{\triangle} \end{pmatrix}$$

which will have a unique solution when the 2×2 matrix is invertible. Namely when the matrix has non-zero determinant.

Definition 2. *[Wronskian]*
For twice delta-differentiable functions y_1 and y_2, we define the Wronskian, $W = W(y_1, y_2)$ by

$$W(t) = \det \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1^{\triangle}(t_0) & y_2^{\triangle}(t_0) \end{pmatrix}$$

Then if $W(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$, y_1, y_2 are a *fundamental set of solutions* to (2).

Theorem 3. *If y_1, y_2 are a fundamental set of solutions to (2), then the solution to (2) and (4) is $y(t) = \alpha y_1(t) + \beta y_2(t)$ where α, β are determined from*

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{W(t_0)} \begin{pmatrix} y_2^\triangle(t_0) & -y_2(t_0) \\ -y_1^\triangle(t_0) & y_1(t_0) \end{pmatrix} \begin{pmatrix} y_0 \\ y_0^\triangle \end{pmatrix} \tag{5}$$

Let us investigate some IVPs applying this theorem. In particular consider the 'constant-coefficient' case

$$y^{\triangle\triangle} + by^\triangle + cy = 0 \tag{6}$$

with $b, c \in \mathbb{R}$, $\mathbb{T}$ arbitrary and (6) regressive. Then $1 - b\mu(t) + c\mu^2(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$. This is equivalent to $1 + (c\mu - b)\mu \neq 0$ for $c\mu - b \in \mathcal{R}$.

To solve (6), it seems natural to try solutions of the form $y(t) = e_\lambda(t, t_0)$ with $\lambda \in \mathcal{R}$ or for λ being constant. Then (6) becomes

$$\begin{aligned} y^{\triangle\triangle}(t) + by^\triangle(t) + cy(t) &= \lambda^2 e_\lambda(t,t_0) + b\lambda e_\lambda(t,t_0) + ce_\lambda(t,t_0) \\ &= \left(\lambda^2 + b\lambda + c\right) e_\lambda(t,t_0) \\ &= 0 \end{aligned}$$

if and only if $\lambda^2 + b\lambda + c = 0$ since $\lambda \in \mathcal{R}$ and $e_\lambda(t,t_0) \neq 0$ on $\mathbb{T}$. We call $\lambda^2 + b\lambda + c = 0$ the characteristic equation which has roots

$$\begin{cases} \lambda_1 = \frac{-b-\sqrt{b^2-4c}}{2} \\ \lambda_2 = \frac{-b+\sqrt{b^2-4c}}{2} \end{cases} \tag{7}$$

Theorem 4. *Let $b^2 - 4c \neq 0$. If $\mu c - b \in \mathcal{R}$, then a fundamental set of solutions to (6) is*

$$e_{\lambda_1}(t,t_0) \quad \& \quad e_{\lambda_2}(t,t_0)$$

Moreover, the solution to (6) and (4) is given by

$$y(t) = \alpha e_{\lambda_1}(t,t_0) + \beta e_{\lambda_2}(t,t_0)$$

where α, β are obtained from (5).

Example 1. *Solve*

$$\begin{cases} y^{\triangle\triangle} + 2y^\triangle - 3y = 0 \\ y(0) = 1 \quad y^\triangle(0) = 2 \end{cases}$$

for $0 \in \mathbb{T}$ and $\mathbb{T}$ arbitrary and $-3\mu - 2 \in \mathcal{R}$.

- *Here $b = 2$ and $c = -3$, so that $b^2 - 4c \neq 0$.*
- *The characteristic equation is $\lambda^2 + 2\lambda - 3 = 0$ and has solutions $\lambda_1 = -3$ and $\lambda_2 = 1$.*

- *Then the general solution is*

$$y(t) = \alpha e_{-3}(t, t_0) + \beta e_1(t, t_0)$$

- *To determine α, β we use (5), so that*

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ \frac{5}{4} \end{pmatrix}$$

Then $y(t) = -\frac{1}{4}e_{-3}(t, 0) + \frac{5}{4}e(t, 0)$ is the solution to this second-order dynamic equation.