Last time: solved \( y' = y, \; y(0) = 1 \) for a general time scale.

Today we want to solve \( y' = p(t) y, \; y(t_0) = 1 \) considering the (more general) dynamic IVP given by
\[
\begin{cases}
y' = p(t) y, \; \forall t \in \mathbb{T}^\kappa \\
y(t_0) = 1
\end{cases}
\]

where \( p: \mathbb{T} \to \mathbb{R}, \; t_0 \in \mathbb{T} \)

Let \( p \in \text{Crd} \) and \( 1 + \mu(t) p(t) \to 0 \) \( \forall t \in \mathbb{T}^\kappa \), then the unique solution to \( \mathcal{J} \) is denoted by:
\( e_p(t, t_0) \)

which is called "the generalized exponential function on \( \mathbb{T} \)."

We can also write \( e_p(t, t_0) \) explicitly:
\[
e_p(t, t_0) = \begin{cases}
\exp \left[ \int_{t_0}^{t} \frac{1}{\mu(r)} \log(1 + \mu(r) p(r)) \, dr \right], \mu > 0 \\
\exp \left[ \int_{t_0}^{t} p(r) \, dr \right], \mu = 0
\end{cases}
\]

Note here that \( \log \) is the principal logarithm.

Note that in the previous lecture, we defined \( e(t, 0) \). We understand this to mean \( e(t, 0) = e_t(0, 0) \).
Define \( p : \mathbb{R} \to \mathbb{R} \) as regressive if \( p \in C_{rd} \) and \( 1 + M(t)p(t) \neq 0 \) for all \( t \in \mathbb{R} \).

The space of all regressive functions is denoted \( \mathcal{R} \) (funny-looking \( R \)).

Some useful properties of \( e_p(t,t_0) \) are summarized in the following theorem:

**Thm.** If \( p, q \in \mathcal{R} \), then

1. \( e_q(t,t_0) = 1 \) and \( e_p(t,t) = 1 \)
2. \( e_p(\sigma(t), t_0) = (1 + M(t)p(t)) e_p(t, t_0) \)
3. \( e_p(t, t_0) = \frac{1}{e_p(t_0, t)} \)
4. \( e_p(t, t_0) e_p(t_0, \tau) = e_p(t, \tau) \)
5. \( \left( \frac{1}{e_p(t, t_0)} \right)^\Delta = -e_p(t) \frac{1}{e_p(t, t_0)} \)

where \( e_p(\sigma(t), t_0) = e_p(\sigma(t), t_0) \)

**Proof (1):** We know \( f^\sigma = f + Mf^\sigma \)

so,

\[
e_p(\sigma(t), t_0) = e_p(t, t_0) + M(t)e_p(\sigma(t), t_0) = e_p(t, t_0) + M(t)e_p(t, t_0)
\]

\[
= (1 + M(t)p(t)) e_p(t, t_0)
\]
Example 1: Consider the IVP
\[
\begin{align*}
y'(t) &= ay, & t \in \mathbb{T}^k, & a \leq -1 \\
y(0) &= 1 & a = \text{constant}
\end{align*}
\]
for \( \mathbb{T} = \mathbb{Z} \), show that the solution to \( \text{E} \) changes sign at every point in \( \mathbb{Z} \).
*What do we mean by 'changes sign'?

Well, a solution, \( y \), to \( \text{E} \) (\( \mathbb{T} = \mathbb{Z} \)) changes sign when \( y(t) \cdot y(\sigma(t)) < 0 \).

Here, \( p(t) = a \), \( t_0 = 0 \).

So our solution:
\[
y(t) = \mathcal{E}_a(t, t_0) = \mathcal{E}_a(t, 0)
\]
and
\[
y(\sigma(t)) = \mathcal{E}_a(\sigma(t), t_0) \\
= \mathcal{E}_a(\sigma(t), 0) \\
= (1 + \mu(t) \alpha) \mathcal{E}_a(t, 0)
\]
by \( \text{THM} \) (11)

So,
\[
y(t) \cdot y(\sigma(t)) = \mathcal{E}_a(t, 0) \cdot \mathcal{E}_a(\sigma(t), 0) \\
= \mathcal{E}_a(t, 0) (1 + \mu(t) \alpha) \mathcal{E}_a(t, 0) \\
= [\mathcal{E}_a(t, 0)]^2 (1 + \alpha) \\
(\alpha = 1, \mathbb{T} = \mathbb{Z})
for general \( T \), we need \( \mu(t) a < 0 \) for our solutions to "oscillate".

Let's compare our solutions to IUP using \( T = \mathbb{Z} \) and \( T = \mathbb{R} \). We already have

\[ y' = ay, \quad a < -1 \]

\[ y(0) = 1 \]

which has solution \( y = e^{at} \).

For \( T = \mathbb{R} \),

\[ y' = ay, \quad a < -1 \]

\[ y(0) = 1 \]

which has solution \( y = e^{at} \).

Let's consider the exponential function in \( \mathbb{Z} \) for general \( a \).

Examples of \( e^{p(t)} \):

(1) \( T = h \mathbb{Z}, h > 0 \)

\[ p(t) = \alpha \in \mathbb{R} \]

then

\[ e^x(t, 0) = (1 + \alpha h)^{t/h} \]

To verify this, we show \( y(t) = e^{x(t, 0)} \) satisfies:

\[ y' = \frac{y(t+h) - y(t)}{h} \]

\[ y' = \alpha y(t) = p(t) y(t) \]

and that \( y(0) = 1 \).
Consider initial condition first:
\[ y(0) = e^{\alpha c}(0,0) = (1+\alpha h)^0 = 1 \]
\[ y^*(t) = \frac{y(t+h)-y(t)}{h} = (1+\alpha h)^{\frac{t+h}{h}} - (1+\alpha h)^{\frac{t}{h}} \]
\[ = (1+\alpha h)^{\frac{t}{h}} (1+\alpha h) - (1+\alpha h)^{\frac{t}{h}} \]
\[ = (1+\alpha h)^{\frac{t}{h}} (1+\alpha h - 1) \]
\[ = \alpha y(t)+ (1+\alpha h)^{\frac{t}{h}} = \alpha y(t) \quad \text{as required.} \]