Last Time: Regulated and $C_{rd}$ function.

Today: More on integration on $\mathbb{T}$.

**Definition:** A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called pre-delta-diff’able on a set $D$ (with region of d’bility $D$) if:

a) $D \subset \mathbb{T}^c$

b) $\mathbb{T}^c \setminus D$ is countable and contains no r-s elements of $\mathbb{T}$

c) $f$ is d-diff’able on $D$

**Example:** If $\mathbb{T}$ have only isolated points then $D = \mathbb{T}^c$.

**Exercise:** If $\mathbb{T} = \mathbb{R}$

$$f = \begin{cases} 
0 & t = 0 \\
\frac{1}{t} & t \in \mathbb{R} \setminus \{0\}
\end{cases}$$

then find $D$.

**Theorem 1:**
If $f : [a, b]_\mathbb{T} \to \mathbb{R}$ (Notes: $[a, b]_\mathbb{T} = [a, b] \cap \mathbb{T}$) is a regulated on then closed and bounded (compact) intervals $[a, b]_\mathbb{T}$ then $f$ is bounded on $[a, b]_\mathbb{T}$.

(i.e. $\exists R \geq 0$ s.t. $\sup_{t \in [a, b]_\mathbb{T}} |f(t)| \leq R$)

**Theorem 2:** (MVT for $\mathbb{T}$)
Let $f, g : \mathbb{T} \to \mathbb{R}$ with both functions pre-d-diff’able on $D$, if

$$|f^\Delta(t)| \leq g^\Delta(t), \quad \forall t \in D$$

then

$$|f(s) - f(r)| \leq g(s) - g(r), \quad \forall r, s \in \mathbb{T}, \; r \leq s$$

**Corollary 1:** Let $f, g$, be pre-d-diff’able with region $D$.

1. If $U$ is a compact interval with end points $r, s \in \mathbb{T}$ then

$$|f(s) - f(r)| \leq \left\{ \sup_{t \in U \cap D} |f^\Delta(t)| \right\} |s - r|$$

2. If $f^\Delta(t) = 0 \forall t \in D$ then $f \equiv$ constant

3. If $f^\Delta(t) = g^\Delta(t) \forall t \in D$ then $f(t) = g(t) + c \forall t \in D$ where $c = \text{constant}$
Proof:

1. Define

\[ g(t) = \left\{ \sup_{\tau \in [r, s]\cap D} |f^\Delta(\tau)| \right\} (t - r), \quad \forall t \in \mathbb{T} \]

then

\[ |f^\Delta(t)| \leq \sup_{\tau \in [r, s]\cap D} |f^\Delta(\tau)| = g^\Delta(t), \quad \forall t \in D \cap [r, s]_D^\kappa \]

So by Theorem 2:

\[ g(t) - g(r) \geq |f(t) - f(r)|, \quad \forall t \in [r, s]_D^\kappa \]

\[ |f(s) - f(r)| \leq g(s) - g(r) = g(s) \]

\[ = \left\{ \sup_{\tau \in [r, s]\cap D} |f^\Delta(\tau)| \right\} (s - r) \]

2. follows from 1.

3. follows from 2.

Theorem 3: (Existence of pre-anti-d-deriv’s)

If \( f : \mathbb{T} \to \mathbb{R} \) is a regulated function then \( \exists \) a function \( F \), which is pre-d-diff’able with region \( D \) s.t.

\[ F^\Delta(t) = f(t), \quad \forall t \in D \]

Definition: Let \( f : \mathbb{T} \to \mathbb{R} \) be regulated. Any function \( F \) as in Theorem 3 is called a pre-anti-delta-derivative of \( f \). We define the indefinite delta integral of \( f \) by:

\[ \int f(t) \Delta t = F(t) + c, \quad c = constant \]

and \( F \) is a pre-anti-delta derivative of \( f \).

We define the Cauchy delta integral by

\[ \int_{r}^{s} f(t) \delta t = F(r) - F(s) \quad \forall r, s \in \mathbb{T} \]

Finally, a function is called anti-delta-derivative of \( f : \mathbb{T} \to \mathbb{R} \) provided \( F^\Delta(t) = f(t) \ \forall t \in \mathbb{T}^\kappa \)
Theorem 4: (Existence of anti-d-derivative)

If \( f \in C_{rd} \) then \( f \) has an anti-d-derivative. In particular, if \( t_0 \in \mathbb{T} \) then

\[
F(t) = \int_{t_0}^{t} f(\tau) \Delta \tau, \quad \forall t \in \mathbb{T}
\]

is an anti-d-derivative of \( f \).

Proof: (Use definition of d-diff’ability)

Let \( f \in C_{rd} \). Then \( f \) is also regulated. Let \( F \) be the function guaranteed to exist by Theorem 3 such that,

\[
F(\tau) = \int_{t_0}^{\tau} f(\omega) \Delta \omega, \quad \forall \tau \in D
\]

We show

\[
F(\tau) = f(\tau) \quad \forall \tau \in D
\]

Obviously,

\[
F(\tau) = f(\tau), \quad \forall \tau \in D
\]

From above so consider \( \tau \in T \setminus D \). The point must be r-d because \( T \setminus D \) cannot contain any r-s points. Since \( \tau \) is r-d and \( f \in C_{rd} \), we must have \( f \) continuous at \( \tau \).

Let \( \epsilon > 0 \), Then \( \exists \) nbd \( U \) of \( t \) with

\[
|f(s) - f(t)| \leq \epsilon \quad \forall s \in U
\]

Let \( h(\tau) = F(\tau) - f(t)(\tau - t_0) \forall \tau \in \mathbb{T} \)

The \( h \) is pre-d-diff’able with region \( D \) and

\[
h(\tau) = F(\tau) - f(t) = f(\tau) - f(t), \quad \forall \tau \in D
\]

Hence

\[
|h(\tau)| \leq |f(s) - f(t)| \leq \epsilon, \quad \forall s \in D \cap U
\]

So

\[
\sup_{s \in D \cap U} |h(\tau)| \leq \epsilon, \quad \forall s \in D \cap U
\]

Hence by Corollary 1, \( \forall r \in U \)

\[
|F(t) - F(r) - f(t)(t - r)| = |h(t) + f(t)(t - t_0) - [h(r) + f(t)(r - t_0)] - f(t)(t - r)|
\]

\[
= |h(t) - h(r)|
\]

\[
\leq \left\{ \sup_{s \in D \cap U} |h(s)| \right\} |t - r|
\]

\[
\leq \epsilon |t - r|
\]

So \( F \) is d-diff’able with \( F(\tau) = f(\tau) \)

Theorem 5:

If \( a, b, c \in \mathbb{T}, \alpha \in \mathbb{R}, f, g \in C_{rd} \).
1. \( \int_{a}^{b} (f(t) + g(t)) \Delta t = \int_{a}^{b} f(t) \Delta t + \int_{a}^{b} g(t) \Delta t \)

2. \( \int_{a}^{b} \alpha f(t) \Delta t = \alpha \int_{a}^{b} f(t) \Delta t \)

3. \( \int_{a}^{b} f(t) \Delta t = - \int_{b}^{a} f(t) \Delta t \)

4. \( \int_{a}^{b} f(t) \Delta t = \int_{a}^{c} f(t) \Delta t + \int_{c}^{b} f(t) \Delta t \)

5. \( \int_{a}^{b} (\sigma(t)) \Delta^\Delta(t) \Delta t = f(b)g(b) - f(a)g(a) - \int_{a}^{b} f\Delta(t)g(t) \Delta t \)

6. \( \int_{a}^{b} f(t) \Delta^\Delta(t) \Delta t = f(b)g(b) - f(a)g(a) - \int_{a}^{b} f\Delta(t)g(\sigma(t)) \Delta t \)

7. \( \int_{a}^{a} f(t) \Delta t = 0 \)