Consider the dynamic IVP with arbitrary $T$.

(1) \[ x^\Delta = f(t, x) \quad t \in [t_0, N]_T \]

(2) \[ x(t_0) = x_0 \quad x_0 \in \mathbb{R}^n, \ t_0 \in \mathbb{R}, \ N \in \mathbb{T} \]

where $f : [t, N]_T \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and solutions to (1) and (2) are of type $x : [t_0, \sigma(N)]_T \to \mathbb{R}^n$ and continuous.

**Question:** If solutions to (1) and (2) exist, when are they unique?

**Theorem 1:** Let $\alpha > 0$ be a constant such that

(3) \[ \| f(t, x_2) - f(t, x_1) \| \leq \alpha \| x_2 - x_1 \| \quad \forall t \in [t_0, N]_T, \ \forall x_1, x_2 \in \mathbb{R}^n \]

(4) \[ \alpha [\sigma(N) - t_0] < 1 \]

then the IVP (1) and (2) has at most one solution.

**Proof:** Let $x_1$ and $x_2$ be two solutions to (1) and (2) and define $u(t) := x_2(t) - x_1(t)$. We write (1) and (2) in the more convenient form

\[ x(t) - x(t_0) = \int_{t_0}^{t} x^\Delta(s) \Delta s, \quad t \in [t_0, \sigma(N)]_T \]

So $x(t) = \int_{t_0}^{t} f(s, x(s)) \Delta s + x_0$ and we must have the following

\[ u(t) = \int_{t_0}^{t} f(s, x_2(s)) - f(s, x_1(s)) \Delta s \]

and hence \[ \|u(t)\| = \|x_2(t) - x_1(t)\| \leq \int_{t_0}^{t} \|f(s, x_2(s)) - f(s, x_1(s))\| \Delta s \]

\[ \leq \int_{t_0}^{\sigma(N)} \alpha \|x_2(t) - x_1(t)\| \Delta s \quad \text{by Lipschitz condition (3)} \]

\[ \leq \alpha \sup_{t \in [t_0, \sigma(N)]} \|x_2(t) - x_1(t)\| (\sigma(N) - t_0) \]

So we get

\[ \sup_{t \in [t_0, \sigma(N)]} \|x_2(t) - x_1(t)\| \leq \alpha \sup_{t \in [t_0, \sigma(N)]} \|x_2(t) - x_1(t)\| (\sigma(N) - t_0) \]

\[ 0 \leq \sup_{t \in [t_0, \sigma(N)]} \|x_2(t) - x_1(t)\| [\alpha (\sigma(N) - t_0) - 1] \]

With (4) it follows that $\sup_{t \in [t_0, \sigma(N)]} \|x_2(t) - x_1(t)\| = 0$ and thus $x_1(t) = x_2(t) \forall t \in [t_0, \sigma(N)]_T$. So if the IVP has solutions, they are unique.

**Example:** Consider

\[ x^\Delta = tx \quad \forall t \in [0, N]_T, \ 0, N \in \mathbb{T} \]

\[ x(0) = x_0 \quad x \in \mathbb{R} \]

Can you find an $\alpha$ such that the Lipschitz condition (3) holds? ($\alpha = N \otimes$)

**Theorem 2:** If $f$ satisfies

(5) \[ 2 \|x_2 - x_1, f(t, x_2) - f(t, x_1)\| + \mu(t) \|f(t, x_2) - f(t, x_1)\|^2 \leq 0, \ \forall t \in [t_0, N]_T, \ \forall x_1, x_2 \in \mathbb{R}^n \]

then the IVP (1) and (2) has at most one solution on $t \in [t_0, N]_T$. 

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Proof: Let $x_1$ and $x_2$ be two solutions to (1) and (2) and $r(t) := \|x_2(t) - x_1(t)\|^2$, $\forall t \in [t_0, \sigma(N)]_T$. From (2) we see, that $r(t_0) = 0$. Now show $r(t)^\Delta \leq 0$. Consider

$$r(t)^\Delta = \langle x_2(t) - x_1(t), x_2(t) - x_1(t) \rangle$$

$$= \langle x_2(t) - x_1(t), f(t, x_1) - f(t, x_2) + \mu(t)\|f(t, x_1) - f(t, x_2)\|^2 \rangle$$

$$\leq 0,$$

by assumption (use product rule and $x^\Delta = x - \mu x^\Delta$).

So we have $r(t_0) = 0$, $r(t) \geq 0$ and $r^\Delta \leq 0 \forall t \in [t, \sigma(N)]_T$. Thus $r = 0$ holds for all $t \in [t, N]_T$. Hence $x_1 = x_2$, $t \in [t, \sigma(N)]_T$.

Remark: In fact you can easily obtain uniqueness of solutions on $[t_0, \infty)_T$.

Theorem 3: If $f$ satisfies

$$2\langle x_2 - x_1, f(t, x_2) - f(t, x_1) \rangle + \mu(t)\|f(t, x_2) - f(t, x_1)\|^2 \leq 0, \forall t \in [t_0, \infty)_T, \forall x_1, x_2 \in \mathbb{R}^n$$

then the IVP (1) and (2) has at most one solution on $t \in [t_0, \infty)_T$.

Theorem 4: If there exists a delta differentiable function $V : \mathbb{R}^n \to [0, \infty)$ such that

$$V(x) = 0 \iff x = 0$$

$$[V(x)]^\Delta \leq 0 \text{ for all } (t, x) \in [t, \infty)_T \times \mathbb{R}^n$$

then there is at most one solution to (1) and (2) on $t \in [t_0, \infty)_T$.

Remark: See that $V(x) = \|x\|^2$ in Theorem 2 and 3 has all these qualities. Notice we have tried to get uniqueness on $[t_0, \infty)_T$ or $[t_0, \sigma(N)]_T$.

Question: Can you extend Theorems 1-4 to cover the case, when we want to gain uniqueness of solutions on $[\rho(K), t_0)_T$ or $(-\infty, t_0)_T$?