Last Time: Consider the 2nd-order dynamic equation

\[ y^{\Delta\Delta} + by^{\Delta} + cy = 0, \quad b, c = \text{constants} \]

when \( b^2 - 4c > 0 \) (and \( c\mu - b \in \mathbb{R} \)) we looked at example of (1)

Today: We partly answer what if \( b^2 - 4c < 0? \)

Consider the simpler case

\[ y^{\Delta\Delta} + \gamma^2 y = 0, \quad \gamma > 0 \text{ constant} \]

The characteristic equation is

\[ \lambda^2 + \gamma^2 = 0 \]

\( \lambda = \pm \gamma i, \ t^2 = -1 \) complex roots.

For this case, the general solution to (1) is

\[ y(t) = A e^{\lambda_1(t, t_0)} + B e^{\lambda_2(t, t_0)} \]

\[ = A e^{i\gamma(t, t_0)} + B e^{-i\gamma(t, t_0)} \]

The appearance of \( i \) and \( -i \) in the general exponential function makes no significant difference to our knowledge of \( e^t \)

\[
\begin{align*}
[e_{i\gamma}(t, t_0)]^\Delta &= i\gamma e_{i\gamma}(t, t_0) \\
[e_{-i\gamma}(t, t_0)]^\Delta &= -i\gamma e_{-i\gamma}(t, t_0) \\
[e_{i\gamma}(t, t_0)]^{\Delta\Delta} &= -\gamma^2 e_{-i\gamma}(t, t_0) \\
[e_{-i\gamma}(t, t_0)]^{\Delta\Delta} &= -\gamma^2 e_{-i\gamma}(t, t_0)
\end{align*}
\]

so (3) is a solution to (2)

Q: How does a solution like (3) compare with a solution to (2) in \( \mathbb{T} = \mathbb{R} \)?

If \( \mathbb{T} = \mathbb{R} \) then (2) becomes

\[ y'' + \gamma^2 y = 0 \]

\[ \lambda = \pm i\gamma \]

\[ = \alpha \pm i\beta \]

we know the general solution is

\[ y(t) = e^{\alpha t}(C \cos \beta t + D \sin \beta t) \]

\[ = C \cos \gamma t + D \sin \gamma t \]
Q: Can we connect (3) and (4) in some natural way?
A: Yes, by introducing a general \( \sin \) and \( \cos \) function

**Definition:** If \( p \in C_{rd} \) and \( \mu p^2 \in \mathbb{R} \) then define \( \cos_p \) and \( \sin_p \) by:

\[
\begin{align*}
\cos_p &= \frac{e^{ip} + e^{-ip}}{2} \\
\sin_p &= \frac{e^{ip} - e^{-ip}}{2i}
\end{align*}
\]

**Lemma:** let \( p \in C_{rd} \), if \( \mu p^2 \in \mathbb{R} \) then

\[
\begin{align*}
\cos^\Delta_p &= -psin_p \\
\sin^\Delta_p &= pcos_p
\end{align*}
\]

**Proof:**

\[
\begin{align*}
\cos^\Delta_p &= \left[ \frac{e^{ip} + e^{-ip}}{2} \right]^\Delta \\
&= p \left[ \frac{i}{2} (e^{ip} + e^{-ip}) \right] \\
&= p \left[ -\frac{1}{2i} (e^{ip} + e^{-ip}) \right] \\
&= -psin_p
\end{align*}
\]

**Remark:** From (5) it is easy to prove a general *Euler’s formula*

\[
e^{ip(t,t_0)} = \cos_p(t,t_0) + isin_p(t,t_0)
\]

**Theorem:** If \( t_0 \in T^\infty \) then

\[
y(t) = C\cos_\gamma(t,t_0) + D\sin_\gamma(t,t_0)
\]

is a general solution to (2).

**Note:** Compare (6) with (4)

**Example:**

1. \( T = \mathbb{Z} \), for \( p = \alpha \) a constant
   
   \( \sin_\alpha(t,0) = ? \), \( \cos_\alpha(t,0) = ? \). Find \( e_\alpha(t,0) \)?
   
   Solve \( y^\Delta = \alpha y \), \( y(0) = 1 \)
   
   For \( T \in \mathbb{Z} \)
   
   \( \Delta y(t) = \alpha y(t), \quad y(0) = 1 \)
Solve recursively

\[ y(t) = (1 + \alpha)^t \]
\[ = e_\alpha(t, 0) \]

From (5)

\[
\sin_\alpha(t, 0) = \frac{e_{i\alpha}(t, 0) - e^{-i\alpha}(t, 0)}{2i} = \frac{(1 + i\alpha)^t - (1 - i\alpha)^t}{2i}
\]
\[
\cos_\alpha(t, 0) = \frac{e_{i\alpha}(t, 0) + e^{-i\alpha}(t, 0)}{2} = \frac{(1 + i\alpha)^t + (1 - i\alpha)^t}{2}
\]

2. If \( T = \mathbb{R} \) then

\[
\sin_\alpha(t, 0) = \sin(\alpha t) \\
\cos_\alpha(t, 0) = \cos(\alpha t)
\]

3. (Exercise)

Find \( e_1(t, 0), \sin_1(t, 0) \) and \( \cos_1(t, 0) \) for \( T = \mathbb{Z}^2 = \{k^2 : k \in \mathbb{Z}\} \)

Q: It seems natural to ask if \( \cos_p \) and \( \sin_p \) satisfy any natural identities?
From Example 1, see that \( \sin^2_p + \cos^2_p \) may not equal 1.

Q: What if \( b^2 - 4ac = 0 \) in (1)? (This will mean roots of characteristics equation are equal).
If \( T = \mathbb{R} \) then we try general solutions of type

\[ y(t) = Ae^{\lambda t} + Bte^{\lambda t} \]

For general time scale case we try solutions of form

\[ y(t) = v(t)e_\lambda(t, t_0) \]

for a suitable function \( v \).