Last time we solved the homogeneous problem $y'' + 2y' - 3y = 0$. What about the inhomogeneous case?

Consider $y'' + 2y' - 3y = f(t)$. If $y_h$ is a solution to the homogeneous problem and if $y_p$ is a particular solution to the inhomogeneous problem then $y = y_h + y_p$.

1. $f(t) \equiv 1$. What is $y_p$? Try $y_p = K$, $K$ a constant. For $y_p = K$, the homogeneous problem becomes $y_p'' + 2y_p' - 3y_p = 0 + 0 - 3K = 1$
   $\Rightarrow K = -\frac{1}{3}$,
   $\Rightarrow y = y_h + y_p = \alpha e^{-3(t,0)} + \beta e(t,0) - \frac{1}{3}, \alpha$ and $\beta$ constants.

2. $f(t) = t$. Try $y_p = At + B$, where $A$ and $B$ are constants. $y_p'' = A, y_p'' = 0$,
   so $y_p'' + 2y_p' - 3y_p = 0 + 2A - 3At - 3B = t$
   $\Rightarrow 2A - 3B = 0$
   $\Rightarrow -3A = 1$, 
   so $A = -\frac{1}{3}, B = \frac{2}{9}$.
   $\Rightarrow y = y_h + y_p = \alpha e^{-3(t,0)} + \beta e(t,0) - \frac{1}{3}t + \frac{2}{9}$

3. $f(t) = t^2$ (open case) Try $y_p = At^2 + Bt + C$, where $A, B, C$ are constants.
   $y_p'' = A[t + \sigma(t)] + B$
   $y_p''$ = ?? (it may not exist because $\sigma$ may not be delta differentiable)

4. $f(t) = e_2(t,0)$. Try $y_p = Ke_2(t,0)$, where $K$ is a constant.
   $y_p'' = 2Ke_2(t,0), y_p'' = 4Ke_2(t,0)$,
   $y_p'' + 2y_p' - 3y_p = 4Ke_2(t,0) + 4Ke_2(t,0) - 3Ke_2(t,0) = e_2(t,0)$
   $\Rightarrow 5K = 1$, so $K = \frac{1}{5}$,
   $\Rightarrow y = y_h + y_p = \alpha e^{-3(t,0)} + \beta e(t,0) + \frac{1}{5}e_2(t,0)$

Consider the system

\[
\begin{align*}
    x' &= 2x + y \\
    y' &= -5x + 4y \\
    x(0) &= 1 \\
    y(0) &= 2
\end{align*}
\]
(2) \( \Rightarrow y^{\Delta \Delta} = -5x^{\Delta} - 4y^{\Delta} \)
\[= -5(2x + y) - 4y^{\Delta} \]
\[= -10x - 5y - 4y^{\Delta} \]
\[= -10\left(\frac{4y - y^{\Delta}}{5}\right) - 5y - 4y^{\Delta} \]
\[= 8y + 2y^{\Delta} - 5y - 4y^{\Delta} \]
\[\Rightarrow y^{\Delta \Delta} + 2y^{\Delta} - 3y = 0 \]
which has solution \( y = Ae_{-3}(t,0) - Be(t,0) \), where \( A \) and \( B \) are constants.

(2) \( \Rightarrow x = -\frac{1}{5}(y^{\Delta} + 4y) \)
\[= -\frac{1}{5}[-3Ae_{-3}(t,0) + Be(t,0) + 4Ae_{-3}(t,0) + 4Be(t,0)] \]
\[= -\frac{1}{5}[Ae_{-3}(t,0) - Be(t,0)] \]
\[= -\frac{1}{5}Ae_{-3}(t,0) - Be(t,0) \]

Now we find \( A \) and \( B \):

\[x(0) = 1 = -\frac{1}{5}Ae_{-3}(0,0) - Be(0,0) \]
\[= -\frac{A}{5} - B \]
\[y(0) = 2 = A + B \]

\[\Rightarrow A = \frac{15}{4}, B = -\frac{7}{4} \]
so our full solution is:

\[x(t) = -\frac{1}{5} \cdot \frac{15}{4} e_{-3}(t,0) + \frac{7}{4} e(t,0) \]
\[= -\frac{3}{4} e_{-3}(t,0) + \frac{7}{4} e(t,0) \]
\[y(t) = \frac{15}{4} e_{-3}(t,0) - \frac{7}{4} e(t,0) \]