Successive approximations to solutions of dynamic equations on time scales

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Abstract

In this paper we establish the method of successive approximations within the field of “dynamic equations on time scales”. Our introduction and application of the method leads to new results concerning the qualitative and quantitative properties of solutions to nonlinear dynamic equations on time scales. The new discoveries include sufficient conditions under which we conclude: existence; uniqueness; and approximation of solutions. We also obtain some comparison results for solutions to dynamic equations; and obtain a relationship between the exponential function on a time scale and the classical exponential function.

Key words: successive approximations; existence and uniqueness of solutions; nonlinear dynamic equations on time scales; initial value problems.

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1 Introduction

The field of “dynamic equations on time scales” first appeared about 20 years ago and is due to Hilger [17] and his supervisor Aulbach. The area provides a general and versatile
mathematical framework from which we gain a deeper understanding of the connections and distinctions between differential equations and difference equations. Moreover, the field of dynamic equations on time scales is of such a general nature that it has the potential to more accurately model certain “hybrid” stop–start processes than either differential equations or difference equations would when used in isolation.

Especially during the past 10 years, there have been a number of significant advances in our understanding of dynamic equations on time scales, including the establishment of a solid foundation of linear theory [6, Chaps.3,4]. However, due to the young age of the field and the difficulties involved, much of the basic nonlinear theory is yet to be established, developed and refined.

In this paper we establish the method of successive approximations within the field of dynamic equations on time scales. Applications of the method provide new results concerning the qualitative and quantitative properties of solutions, including: existence; uniqueness; and approximation of solutions.

The method of successive approximations is a very powerful tool that dates back to the works of Liouville [23, p.444] and Picard [27]. The method involves endeavouring to solve an equation of type

\[ y = F(y); \]  

where \( F \) is continuous, by starting from some \( y_0 \) and then defining a sequence \( y_n \) of approximations by

\[ y_{n+1} := F(y_n), \quad n = 1, 2, \ldots \]

If \( y_n \) converges to some \( y^* \) then \( y^* \) will, in fact, be a solution to (1).

We will consider the following dynamic initial value problem (IVP)

\[ x_\Delta = f(t, x), \quad t \in J^\kappa; \]  
\[ x(t_0) = x_0; \]

where: \( J^\kappa \subseteq J \) with \( J \) being a closed (possibly unbounded above) subset of a “time scale” \( T \) (which is a closed, nonempty subset of the real numbers) ; with the function \( f \) mapping from \( J^\kappa \times D \subseteq J^\kappa \times \mathbb{R} \) to \( \mathbb{R} \); and \((t_0, x_0)\) is a given point in \( J^\kappa \times D \). The \( \Delta \) denotes “delta differentiation”, which will be more precisely defined a little later.

We organise this work as follows. In Section 2 we present the notation and definitions of dynamic equations on time scales.

In Section 3 we establish some preliminary results concerning the “nonmultiplicity” of solutions to (2), (3) – that is, when the problem admits, at most, one solution.

Then, in Section 4 the method of successive approximations is applied to (2), (3) to obtain local and nonlocal existence, uniqueness and approximation results for solutions.

In Section 5 we discuss whether Hilger continuity of \( f \) alone is sufficient to guarantee the convergence of the sequence (or a subsequence) of successive approximations by constructing an interesting example.

We further investigate the approximation of solutions to (2), (3) in Section 6 by establishing some bounds and convergence properties on solutions to approximative problems.

In Section 7 we use the method of successive approximations to obtain some comparison results between the generalised exponential function on a time scale and the classical exponential function.
Finally, in Section 8 we include some generalisations of our results and in Section 9 we propose some future directions.

In [32] some basic existence, uniqueness and approximation results for (2), (3) were developed via various fixed–point methods and topological ideas, including Banach’s fixed–point theorem and Schäfer’s fixed–point theorem. In contrast to the methods used in [32], the successive approximation approach used herein provides a significant advantage in that the proofs rely on ideas from classical analysis – a knowledge of metric spaces and functional analysis is not required. Furthermore, in new results such as: Theorem 4.10; Remark 4.11; Theorem 8.2; and Remark 8.3, Banach’s fixed–point theorem is unavailable for use in the respective proofs.

We believe this work, [19] and [32] provide a platform from which to launch advanced studies into the field of nonlinear dynamic equations on time scales.

For additional works on dynamic equations on time scales, the reader is referred to [1, 2, 3, 7, 8, 13, 18, 20, 25, 26, 30] and references therein.

2 Preliminaries

To understand the notation used above and to keep the paper somewhat self–contained, this section contains some preliminary definitions and associated notation. For more detail see [6, Chap.1] or [17].

Definition 2.1 A time scale \( T \) is a nonempty closed subset of the real numbers \( \mathbb{R} \).

Since a time scale may or may not be connected, the concept of the jump operator is useful to define the generalised derivative \( x^\Delta \) of a function \( x \).

Definition 2.2 The forward (backward) jump operator \( \sigma(t) \) at \( t < \sup T \) (respectively \( \rho(t) \) at \( t > \inf T \)) is given by
\[
\sigma(t) := \inf \{ \tau > t : \tau \in T \}, \quad (\rho(t) := \sup \{ \tau < t : \tau \in T \}) \text{, for all } t \in T.
\]
Define the graininess function \( \mu : T \to [0, \infty) \) as \( \mu(t) := \sigma(t) - t \).

Throughout this work the assumption is made that \( T \) has the topology that it inherits from the standard topology on the real numbers \( \mathbb{R} \).

Definition 2.3 The jump operators \( \sigma \) and \( \rho \) allow the classification of points in a time scale in the following way: If \( \sigma(t) > t \), then the point \( t \) is called right–scattered; while if \( \rho(t) < t \), then \( t \) is termed left–scattered. If \( t < \sup T \) and \( \sigma(t) = t \), then the point \( t \) is called right–dense; while if \( t > \inf T \) and \( \rho(t) = t \), then we say \( t \) is left–dense.

If \( T \) has a left–scattered maximum value \( m \), then we define \( T^c := T - \{ m \} \). Otherwise \( T^c := T \).

The following gives a formal \( \varepsilon – \delta \) definition of the generalised delta derivative.
Definition 2.4 Fix $t \in \mathbb{T}^c$ and let $x : \mathbb{T} \to \mathbb{R}$. Define $x^\Delta(t)$ to be the number (if it exists) with the property that given $\varepsilon > 0$ there is a neighbourhood $U$ of $t$ with
\[ |[x(\sigma(t)) - x(s)] - x^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|, \quad \text{for all } s \in U. \]

Call $x^\Delta(t)$ the delta derivative of $x(t)$ and say that $x$ is delta–differentiable.

If $\mathbb{T} = \mathbb{R}$ then $x^\Delta = x'$, while if $\mathbb{T} = \mathbb{Z}$ then $x^\Delta = \Delta x$.

Converse to the delta derivative, we now state the definition of the delta integral.

Definition 2.5 If $K^\Delta(t) = k(t)$ then define the delta integral by
\[ \int_a^t k(s) \Delta s = K(t) - K(a). \]

If $\mathbb{T} = \mathbb{R}$ then $\int_a^t k(s) \Delta s = \int_a^t k(s) ds$, while if $\mathbb{T} = \mathbb{Z}$ then $\int_a^t k(s) \Delta s = \sum_{s=a}^{t-1} k(s)$. Once again, there are many more time scales than just $\mathbb{R}$ and $\mathbb{Z}$ and hence there are many more delta integrals. For a more general definition of the delta integral see [6].

The following theorem will be very useful.

Theorem 2.6 [17] Assume that $k : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^c$.

(i) If $k$ is delta–differentiable at $t$ then $k$ is continuous at $t$.

(ii) If $k$ is continuous at $t$ and $t$ is right–scattered then $k$ is delta–differentiable at $t$ with
\[ k^\Delta(t) = \frac{k(\sigma(t)) - k(t)}{\sigma(t) - t}. \]

(iii) If $k$ is delta–differentiable at $t$ and $t$ is right–dense then
\[ k^\Delta(t) = \lim_{s \to t^-} \frac{k(t) - k(s)}{t - s}. \]

(iv) If $k$ is delta–differentiable at $t$ then $k(\sigma(t)) = k(t) + \mu(t)k^\Delta(t)$.

For brevity, we will write $x^\sigma$ to denote the composition $x \circ \sigma$.

The following gives a generalised idea of continuity on time scales.

Definition 2.7 Assume $k : \mathbb{T} \to \mathbb{R}$. Define and denote $k \in C_{rd}(\mathbb{T})$ as right–dense continuous (rd–continuous) if: $k$ is continuous at every right–dense point $t \in \mathbb{T}$; and $\lim_{s \to t^-} k(s)$ exists and is finite at every left–dense point $t \in \mathbb{T}$.

Of particular importance is the fact that every $C_{rd}$ function is delta–integrable [6, Theorem 1.74].

We will also consider (2) with a Hilger continuous right–hand side. This definition is more general than the usual assumption of continuity. The mapping $f : [a, b]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is called Hilger continuous if: $f$ is continuous at each $(t, x)$ where $t$ is right–dense; and the limits
\[ \lim_{(s,y) \to (t^- , x)} f(s, y) \quad \text{and} \quad \lim_{y \to x} f(t, y) \]
both exist (and are finite) at each \((t, x)\) where \(t\) is left–dense. We have introduced this particular term for functions of several variables, to avoid confusion with right–dense continuous functions of one variable.

Define the so–called set of regressive functions, \(\mathcal{R}\), by
\[
\mathcal{R} := \{ p \in C_{rd}(\mathbb{T}; \mathbb{R}) \text{ and } 1 + p(t)\mu(t) \neq 0, \text{ for all } t \in \mathbb{T} \};
\]
and the set of positively regressive functions, \(\mathcal{R}^+\), by
\[
\mathcal{R}^+ := \{ p \in C_{rd}(\mathbb{T}; \mathbb{R}) \text{ and } 1 + p(t)\mu(t) > 0, \text{ for all } t \in \mathbb{T} \}.
\]
For \(p \in \mathcal{R}\) we define (see [6, Theorem 2.35]) the exponential function \(e_p(\cdot, t_0)\) on the time scale \(\mathbb{T}\) as the unique solution to the scalar IVP
\[
x^\Delta = p(t)x, \quad x(t_0) = 1.
\]
If \(p \in \mathcal{R}^+\) then \(e_p(t, t_0) > 0\) for all \(t \in \mathbb{T}\), [6, Theorem 2.48].

More explicitly, the exponential function \(e_p(t, t_0)\) is given by
\[
e_p(t, t_0) := \begin{cases} 
\exp \left( \int_{t_0}^t p(s) \, ds \right), & \text{for } t \in \mathbb{T}, \mu = 0; \\
\exp \left( \int_{t_0}^t \log(1 + \mu(s)p(s)) \frac{\Delta s}{\mu(s)} \right), & \text{for } t \in \mathbb{T}, \mu > 0;
\end{cases}
\]
where \(\log\) is the principal logarithm function.

**Definition 2.8** By a solution to the dynamic IVP (2) we mean a delta–differentiable function \(\phi : I \subseteq J \rightarrow \mathbb{R}\) such that the points \((t, \phi(t))\) lie in \(I \times D\) for all \(t \in I \subseteq J\) and \(\phi(t)\) satisfies (2) for all \(t \in I^\kappa \subseteq J^\kappa\).

### 3 Nonmultiplicity of solutions

In this section we obtain some new results concerning the nonmultiplicity of solutions to the dynamic IVP (2), (3). In particular, we will provide sufficient conditions under which the IVP will have, at most, one solution – that is, there will either be one solution to the problem, or there will be no solution at all. The results will be of use in some of the main proofs in latter sections.

The following result is known as Grönwall’s inequality on time scales [6, Theorem 6.4] and will be used throughout this work.

**Theorem 3.1 (Grönwall)** Let \(y, z \in C_{rd}\) and \(q \in \mathcal{R}^+\) with \(q \geq 0\). If
\[
y(t) \leq z(t) + \int_{t_0}^t q(s)y(s) \Delta s, \quad \text{for all } t \in \mathbb{T};
\]
then
\[
y(t) \leq z(t) + e_q(t, t_0) \int_{t_0}^t z(s)q(s) \frac{1}{e_q(\sigma(s), t_0)} \Delta s, \quad \text{for all } t \in \mathbb{T}.
\]
We now present our main result concerning the nonmultiplicity of solutions to (2), (3). The result is of importance in its own right, but will also be applied later in this paper.

**Theorem 3.2** Let \( f : J^\kappa \times D \subseteq J^\kappa \times \mathbb{R} \to \mathbb{R} \) be Hilger continuous. If there exists a right–dense continuous function \( l : J^\kappa \to [0, \infty) \) such that \( f \) satisfies

\[
|f(t, p) - f(t, q)| \leq l(t)|p - q|, \quad \text{for all } (t, p), (t, q) \in J^\kappa \times D;
\]

then there is, at most, one solution \( x \) to the dynamic IVP (2), (3) such that \( x(t) \in D \) for all \( t \in J \).

**Proof:** Let \( x \) and \( y \) be two solutions such that \( x(t) \in D \) and \( y(t) \in D \) for all \( t \in J \). Since \( f \) is Hilger continuous, it follows from [32, Lemma 2.3] that

\[
x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) \, \Delta s, \quad \text{for all } t \in J;
\]

with a similar well–defined delta integral equation holding for \( y \). We show that \( x \equiv y \) on \( J \).

For each \( t \in J \), consider

\[
r(t) := |x(t) - y(t)|
\]

\[
\leq \int_{t_0}^{t} |f(s, x(s)) - f(s, y(s))| \, \Delta s
\]

\[
\leq \int_{t_0}^{t} l(s)|x(s) - y(s)| \, \Delta s
\]

\[
= \int_{t_0}^{t} l(s)r(s) \, \Delta s.
\]

Thus, applying Grönwall’s inequality we see that \( r \equiv 0 \) on \( J \) and the result follows. \( \square \)

4 **Successive approximations**

In this section we establish and apply the method of successive approximations within the setting of dynamic equations on time scales. Consequently, we uncover new techniques and results concerning the existence, uniqueness and approximation of solutions to the dynamic IVP (2), (3).

We define a sequence of functions \( \phi_0, \phi_1, \ldots \) each defined on some subset of \( J \) in the following way:

\[
\phi_0(t) \equiv x_0,
\]

\[
\phi_{k+1}(t) := x_0 + \int_{t_0}^{t} f(s, \phi_k(s)) \, \Delta s, \quad k = 1, 2, \ldots .
\]

We will formulate some conditions on \( f \) and produce some interval \( I \subseteq J \) such that the sequence of functions \( \phi_k \) converge uniformly on \( I \) to a function \( \phi \), with \( \phi \) being a solution to the IVP (2), (3). In addition, we show that the \( \phi_k \) will successively approximate the solution \( \phi \) in the sense that the “error” between the two monotonically decreases as \( k \) increases.
4.1 Picard–Lindelöf result

Let $J := [t_0, t_0 + a] \subseteq [t_0, t_0 + a] \cap \mathbb{T}$, where $a > 0$ is a constant and $t_0, t_0 + a \in \mathbb{T}$. Let $R^c$ be the rectangle defined by

$$ R^c := \{(t, p) \in \mathbb{T}^c \times \mathbb{R} : t \in [t_0, t_0 + a], |p - x_0| \leq b\}, \tag{8} $$

where $b > 0$ is a constant. Similarly, let $R$ be the rectangle defined by

$$ R := \{(t, p) \in \mathbb{T} \times \mathbb{R} : t \in [t_0, t_0 + a], |p - x_0| \leq b\}. \tag{9} $$

Consider (2), (3) with $f : R^c \to \mathbb{R}$ being Hilger continuous. Our objective is to formulate conditions, inspired by the method of successive approximations, under which there exists some interval $I \subseteq [t_0, t_0 + a] \cap \mathbb{T}$, where $t_0, t_0 + a \in \mathbb{T}$ such that there exists a unique solution $\phi$ to (2), (3). We achieve this by proving a Picard–Lindelöf theorem in the time scale environment.

Since $f$ is Hilger continuous on the compact rectangle $R^c$, there exists a constant $M > 0$ such that

$$ |f(t, p)| \leq M, \text{ for all } (t, p) \in R^c. \tag{10} $$

Once $M$ is determined from $f$ and $R^c$, we will choose the interval $I$ to be $I := [t_0, t_0 + \alpha] \subseteq [t_0, t_0 + a] \cap \mathbb{T}$, where

$$ \alpha := \min\left\{ a, \frac{b}{M} \right\}. \tag{11} $$

The above choice of $\alpha$ is natural for two reasons. Firstly, the condition $\alpha \leq a$ is necessary. Secondly, the restriction $\alpha \leq b/M$ is governed by the fact that if $x = x(t)$ is a solution to (2) on $I = [t_0, t_0 + \alpha]$, then for each $t \in I$ we have

$$ |x(t) - x_0| \leq \int_{t_0}^{t} |x^\Delta(s)| \, \Delta s \leq M(t - t_0) \leq M\alpha. $$

thus the graph of $(t, x(t))$ will lie in the rectangle $R$ for all $t \in I$.

It may happen that $\mathbb{T}$, $f$ and $R$ are such that $I$ contains only the point $t_0$. We avoid this situation by assuming throughout that $[t_0, \sigma(t_0)] \subset I$.

**Lemma 4.1** For each $k$, the successive approximations $\phi_k$ defined in (6), (7) exist as continuous functions on $I$ and the points $(t, \phi_k(t))$ lie in the rectangle $R$ for each $k$ and all $t \in I$. In fact, each $\phi_k$ satisfies

$$ |\phi_k(t) - x_0| \leq M(t - t_0), \text{ for all } t \in I. \tag{12} $$

**Remark 4.2** Inequality (12) implies that each $\phi_k$ satisfies

$$ |\phi_k(t) - x_0| \leq b, \text{ for all } t \in I; $$

which illustrates that the points $(t, \phi_k(t))$ lie in the rectangle $R$ for each $k$ and all $t \in I$. 

Proof of Lemma 4.1: We use proof by induction. It is easy to see that $\phi_0$ is defined as a continuous function on $I$ and that (12) holds for $k = 0$. Now assume that for some $k = n \geq 0$, $\phi_n$ is defined as a continuous function on $I$ and satisfies (12). Since $f$ is Hilger continuous on $R^c$ and $(t, \phi_n(t)) \in R$ we see that (7) is well–defined on $I$. Furthermore, from (7) we see that $\phi_{n+1}$ will be a continuous function on $I$. Also, for all $t \in I$ we have

$$|\phi_{n+1}(t) - x_0| = \left| \int_{t_0}^t f(x, \phi_n(s)) \Delta s \right| \leq \int_{t_0}^t |f(x, \phi_n(s))| \Delta s \leq M(t - t_0) \leq M\alpha \leq b.$$ 

Thus, (12) holds for $k = n + 1$ and so (12) must hold for all integers $k \geq 0$ by induction. □

The following lemma comes from [22, Theorem 2.3] (see also [4, Lemma 2.1]) and will be very useful in the proofs of our main results. The lemma provides a way of estimating certain delta integrals in terms of regular integrals.

Lemma 4.3 Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous, nondecreasing function. If $\lambda_1, \lambda_2 \in \mathbb{T}$ with $\lambda_1 \leq \lambda_2$ then

$$\int_{\lambda_1}^{\lambda_2} h(t) \Delta t \leq \int_{\lambda_1}^{\lambda_2} h(t) dt.$$ 

We now present the Picard–Lindelöf theorem for dynamic equations on time scales.

Theorem 4.4 Let $f$ be Hilger–continuous on the rectangle $R^c$ in (8) and let $\alpha$ be defined in (11). Suppose there exists a constant $L > 0$ such that $f$ satisfies

$$|f(t, p) - f(t, q)| \leq L|p - q|, \text{ for all } (t, p), (t, q) \in R^c.$$ 

Then the successive approximations $\phi_k$ defined in (6), (7) converge uniformly on $I := [t_0, t_0 + \alpha]_\mathbb{T}$ to the unique solution of (2), (3).

Remark 4.5 Note that Theorem 4.4 is an existence and uniqueness result concerning solutions to the dynamic IVP (2), (3).

Proof of Theorem 4.4: For expository purposes we split the proof up into different parts.

1. Uniform convergence of $\phi_k$ on $I$:

   We write $\phi_k$ as the following sum

   $$\phi_k = \phi_0 + (\phi_1 - \phi_0) + (\phi_2 - \phi_1) + \ldots + (\phi_k - \phi_{k-1}),$$

   so

   $$\lim_{k \to \infty} \phi_k(t) = \phi_0(t) + \sum_{i=1}^{\infty} (\phi_i(t) - \phi_{i-1}(t)), \text{ for all } t \in I;$$

   (14)
provided that (14) converges. Thus, showing that the sequence of functions \( \phi_k \) converges uniformly on \( I \) is equivalent to showing that the right–hand side of (14) converges uniformly on \( I \). To prove the latter we establish an estimate on the terms \( |\phi_i(t) - \phi_{i-1}(t)| \) for each integer \( i \geq 1 \) and all \( t \in I \) and then employ Weierstrass’ test for uniformly convergent series [12, p.201].

For each integer \( i \geq 1 \) we prove the following estimate by induction

\[
|\phi_i(t) - \phi_{i-1}(t)| \leq \frac{M}{L} \frac{L^i(t - t_0)^i}{i!}, \quad \text{for all } t \in I. \quad (15)
\]

For \( i = 1 \) see that (15) holds from (12) in Lemma 4.1. Now assume that (15) holds for some \( i = n \geq 1 \). See for all \( t \in I \) we then have

\[
|\phi_{n+1}(t) - \phi_n(t)| \leq \int_{t_0}^t |f(s, \phi_n(s)) - f(s, \phi_{n-1}(s))| \, ds \\
\leq L \int_{t_0}^t |\phi_n(s) - \phi_{n-1}(s)| \, ds \\
\leq L \int_{t_0}^t \frac{M L^n(s - t_0)^n}{n!} \, ds \\
\leq \frac{M L^{n+1}}{L} \int_{t_0}^t \frac{(s - t_0)^n}{n!} \, ds \\
= \frac{M L^{n+1}(t - t_0)^{n+1}}{L} \frac{(n + 1)!}{(n + 1)!},
\]

where we have employed (13) and Lemma 4.3 above. Thus, (15) holds for \( i = n + 1 \) and so (15) holds for all integers \( i \geq 1 \) by induction.

For each integer \( i \geq 1 \) and all \( t \in I \) the estimate (15) then gives

\[
|\phi_i(t) - \phi_{i-1}(t)| \leq \frac{M (L \alpha)^i}{L} \frac{1}{i!}.
\]

We see that

\[
\sum_{i=1}^{\infty} \frac{M (L \alpha)^i}{L} \frac{1}{i!} = \frac{M}{L} \left(-1 + \sum_{i=0}^{\infty} \frac{(L \alpha)^i}{i!}\right),
\]

which converges (to \( M[e^{L \alpha} - 1]/L \)). This convergence enables the application of Weierstrass’ test to conclude that the series

\[
\sum_{i=1}^{\infty} |(\phi_i(t) - \phi_{i-1}(t))|
\]

converges uniformly for all \( t \in I \). The absolute uniform convergence of (16) on \( I \) implies that the right–hand side of (14) converges uniformly for all \( t \in I \) to some function \( \phi(t) \) for all \( t \in I \). That is, there exists a function \( \phi \) on \( I \) such that

\[
\lim_{k \to \infty} \phi_k(t) = \phi(t), \quad \text{for all } t \in I. \quad (17)
\]
2. The limit function $\phi$ is a solution to (2), (3):

We show that: $\phi$ is continuous on $I$; the graph $(t, \phi(t))$ lies in $R$ for all $t \in I$; and that $\phi$ satisfies

$$\phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s)) \Delta s, \text{ for all } t \in I. \quad (18)$$

From (7) we have for all $t, t_1 \in I$

$$|\phi_{k+1}(t) - \phi_{k+1}(t_1)| = \left| \int_{t_1}^{t} f(s, \phi_k(s)) \Delta s \right| \leq M |t - t_1|. \quad (19)$$

Taking limits on the left–hand side of (19) as $k \to \infty$ we obtain

$$|\phi(t) - \phi(t_1)| \leq M |t - t_1|, \text{ for all } t, t_1 \in I. \quad (20)$$

Thus, from (20), for each $\epsilon > 0$ can choose $\delta(\epsilon) := \epsilon / M$ such that

$$|\phi(t) - \phi(t_1)| \leq \epsilon, \text{ whenever } t \in (t_1 - \delta, t_1 + \delta)_T.$$

Thus, $\phi$ is continuous on $I$.

Replacing $t_1$ with $t_0$ in (20) we see that for all $t \in I$

$$|\phi(t) - \phi(t_0)| = |\phi(t) - x_0| \leq M(t - t_0),$$

so that the graph $(t, \phi(t))$ lies in $R$ for all $t \in I$ from Remark 4.2.

We show that $\phi$ satisfies (18) on $I$. See that the right–hand side of (18) is well–defined as $f$ is Hilger continuous on $R^\kappa$ and we have just shown that the graph $(t, \phi(t)) \in R$ for all $t \in I$. Comparing (18) with (7), we know that the left–hand side of (7) converges uniformly on $I$, so all that remains to be shown is

$$\lim_{k \to \infty} \int_{t_0}^{t} f(s, \phi_k(s)) \Delta s = \int_{t_0}^{t} f(s, \phi(s)) \Delta s, \text{ for all } t \in I. \quad (21)$$

Since $\phi_k$ converges uniformly to $\phi$ on $I$, the (uniform) Hilger continuity of $f$ on $R^\kappa$ ensures that

$$\lim_{k \to \infty} f(t, \phi_k(t)) = f(t, \lim_{k \to \infty} \phi_k(t)) = f(t, \phi(t))$$

with the convergence being uniform on $I$.

Thus, an application of [17, Theorem 4.3 (iv)] gives (21).

3. The solution $\phi$ is unique:

Since (13) holds, we know from Theorem 3.2 that (2), (3) can have, at most, one solution. Thus the solution $\phi$ established above is the only solution to (2), (3).
Remark 4.6 Theorem 4.4 could have been proved using Banach’s fixed–point theorem, however, one significant advantage of our proof is that it only relies on ideas from classical analysis and explicitly illustrates the ideas involved – a knowledge of metric spaces and functional analysis is not required.

The following result concerns how well on $I$ the $k$–th approximation $\phi_k$ approximates the solution $\phi$ in Theorem 4.4.

**Theorem 4.7** Under the conditions of Theorem 4.4 the $k$–th approximation $\phi_k$ approximates the solution $\phi$ to the dynamic IVP (2), (3) in the following sense: For $k = 0, 1, \ldots$ we have

$$|\phi(t) - \phi_k(t)| \leq \frac{M (L^\alpha)^{k+1}}{L (k+1)!} e^{L^\alpha}, \quad \text{for all } t \in I.\quad (22)$$

**Proof:** For all $t \in I$ we have

$$\phi_k(t) = \phi_0(t) + \sum_{i=1}^{k} (\phi_i(t) - \phi_{i-1}(t)), \quad k = 0, 1, 2, \ldots$$

$$\phi(t) = \phi_0(t) + \sum_{i=1}^{\infty} (\phi_i(t) - \phi_{i-1}(t)),$$

where we have used (17). Thus, for all $t \in I$ and each $k \geq 0$

$$|\phi(t) - \phi_k(t)| \leq \sum_{i=k+1}^{\infty} |\phi_i(t) - \phi_{i-1}(t)|$$

$$\leq \frac{M}{L} \sum_{i=k+1}^{\infty} \frac{(L^\alpha)^i}{i!}$$

$$\leq \frac{M (L^\alpha)^{k+1}}{L (k+1)!} \sum_{i=0}^{\infty} \frac{(L^\alpha)^i}{i!}$$

$$= \frac{M (L^\alpha)^{k+1}}{L (k+1)!} e^{L^\alpha},$$

where we have used (15). Thus, (22) holds. □

Remark 4.8 To establish (21) the estimate (22) could have been established within the proof of Theorem 4.4 and applied therein, instead of appealing to [17, Theorem 4.3 (iv)] (which concerns the convergence of delta integrals of sequences of functions). To see this, consider for all $t \in I$ and each $k = 0, 1, \ldots$,

$$\left| \int_{t_0}^{t} f(s, \phi(s)) - f(s, \phi_k(s)) \Delta s \right| \leq L \int_{t_0}^{t} |\phi(s) - \phi_k(s)| \Delta s$$

$$\leq \frac{M (L^\alpha)^{k+1}}{L (k+1)!} e^{L^\alpha}.\quad (23)$$
Taking limits as \( k \to \infty \) we see that the right-hand side of (23) tends to zero and thus this establishes (21).

We now provide an example to illustrate an application of Theorem 4.4.

**Example 4.9** Consider the dynamic IVP

\[
\begin{align*}
    x^\Delta & = \sigma(t) + x^2, \quad t \in [0,1]_T; \\
    x(0) & = 0.
\end{align*}
\]  

(24)  

We claim that this IVP has a unique solution on \([0,2/5]_T\).

**Proof:** Let \( f(t,p) := \sigma(t) + p^2 \). We will apply Theorem 4.4 to (24), (25). From (24) we have \( a = 1 \) and we construct the rectangle \( R \) by choosing \( b = 2 \). Then, we see \( M = 5 \) and \( \alpha = 2/5 \).

For any continuous function \( y : [0,1]_T \to \mathbb{R} \), it follows that \( f(t,y(t)) = \sigma(t) + |y(t)|^2 \) is rd–continuous for each \( t \in [0,1]_T \). Thus, the right-hand side of (24) is Hilger continuous on our \( R \) (see [6, Definition 8.14]).

To show that \( f \) satisfies a Lipschitz condition (13) for some \( L > 0 \) on \( R^\kappa \), we show that \( \partial f/\partial p \) is bounded and continuous on \( R^\kappa \). This bound will then be the desired Lipschitz constant \( L \) – see [9, p.208].

We see that for all \((t,p) \in R^\kappa\) we have

\[
\left| \frac{\partial f}{\partial p}(t,p) \right| = |2p| \leq 4.
\]

Also, see that \( \partial f / \partial p \) is continuous on \( R^\kappa \). Thus, (13) holds for \( L = 4 \).

All of the conditions of Theorem 4.4 are satisfied and so the claim follows. \( \square \)

### 4.2 Other local existence results

The following result uses monotonicity of \( f \), rather than a Lipschitz condition, but does not necessarily ensure uniqueness of solutions.

**Theorem 4.10** Let \( f \) be Hilger continuous on the rectangle \( R^\kappa \) in (8) and let \( \alpha \) be defined in (11). If

\[
\begin{align*}
    f(t,p) & \leq f(t,q), \quad \text{for all } p \leq q, \text{ and all } t \in J^\kappa; \\
    f(t,x_0) & \leq 0, \quad \text{for all } t \in J^\kappa,
\end{align*}
\]  

(26)  

(27)

then the successive approximations \( \phi_k \) defined in (6), (7) converge uniformly on the interval \( I := [t_0,t_0 + \alpha]_T \) to a solution of the dynamic IVP (2), (3).

**Proof:** Once again we divide the proof into smaller parts for the sake of clarity.

1. **Uniform convergence of \( \phi_k \) on \( I \):**

   We show that on the interval \( I \), for each \( k \), the \( \phi_k \) defined in (6), (7) are: uniformly bounded; nonincreasing; and equicontinuous. Under these circumstances [21, p.725] the \( \phi_k \) will converge uniformly in \( I \) to a function \( \phi \).
To show that the graph of each \((t, \phi_k(t))\) lies in \(R\), and so each \(\phi_k\) is uniformly bounded on \(I\), it is just a matter of repeating the relevant induction steps as in the proof of Theorem 4.4. Thus, this part is omitted.

Now we show the monotonicity of \(\phi_k\), that is, \(\phi_{k+1}(t) \leq \phi_k(t)\) for each integer \(k \geq 0\) and all \(t \in I\). We use proof by induction. For all \(t \in I\) consider

\[
\phi_1(t) := x_0 + \int_{t_0}^{t} f(s, \phi_0(s)) \Delta s \\
= x_0 + \int_{t_0}^{t} f(s, x_0) \Delta s \\
\leq x_0 \\
= \phi_0(t),
\]

where we have used (27).

Now assume that \(\phi_{n+1}(t) \leq \phi_n(t)\) for some \(k = n \geq 0\) and all \(t \in I\). See, for all \(t \in I\), that

\[
\phi_{n+2}(t) := x_0 + \int_{t_0}^{t} f(s, \phi_{n+1}(s)) \Delta s \\
\leq x_0 + \int_{t_0}^{t} f(s, \phi_n(s)) \Delta s \\
= \phi_{n+1}(t),
\]

where we used (26) above.

Thus, by induction, we see that the sequence \(\phi_k\) is monotone nonincreasing on \(I\).

To show the equicontinuity of \(\phi_k\) on \(I\), it is just a matter of repeating the relevant part of the proof of Theorem 4.4. We obtain, for all \(t, t_1 \in I\)

\[
|\phi_{k+1}(t) - \phi_{k+1}(t_1)| \leq M|t - t_1|.
\]

Thus, the \(\phi_k\) are Lipschitz continuous on \(I\) and so they are equicontinuous.

The above arguments ensure that the \(\phi_k\) converge uniformly on \(I\) to a function \(\phi\).

2. The limit function \(\phi\) is a solution to (2), (3):

Once again, this follows the same arguments as used in the proof of Theorem 4.4. Thus we omit the details.

\[\square\]

**Remark 4.11** In Theorem 4.10 the condition (27) may be replaced by the weaker condition

\[
\int_I f(s, x_0) \Delta s \leq 0,
\]

with the conclusion of Theorem 4.10 still holding. No major changes in the proof are required. This relaxation allows the treatment of IVPs where \(f(t, x_0)\) may change sign for \(t \in J\).
Example 4.12 Consider the dynamic IVP
\[ x^\Delta = x^3, \quad t \in [0, 2]^\mathbb{T}; \]  
\[ x(0) = -1. \]  
(28)  
(29)
We claim that this IVP has at least one solution on \([0, 1/4]^\mathbb{T}\).

Proof: Let \(f(t, p) := p^3\). We will apply Theorem 4.10 to (28), (29). From (28) we have \(a = 2\) and we construct the rectangle \(R\) by choosing \(b = 2\). Then, we see \(M = 8\) and \(\alpha = 1/4\).

For any continuous function \(y : [0, 2]^\mathbb{T} \rightarrow \mathbb{R}\), it follows that \(f(t, y(t)) := [y(t)]^3\) is rd–continuous for each \(t \in [0, 2]^\mathbb{T}\). Thus, the right–hand side of (28) is Hilger continuous on our \(R^\mathbb{C}\).

For \(p \leq q\) it follows that \(p^3 \leq q^3\) and so \(f\) satisfies (26). Furthermore, it is clear that (27) holds.

All of the conditions of Theorem 4.10 are satisfied and so the claim follows. \(\square\)

4.3 Nonlocal results

Theorems 4.4 and 4.10 are called local existence theorems because they only guarantee the existence of a solution \(x(t)\) that is defined for points \(t \in \mathbb{T}\) that lie “close to” the initial point \(t_0\). However, in many situations a solution will actually exist on the entire interval \(J := [t_0, t_0 + a]^\mathbb{T}\).

By revising the proof of Theorem 4.4 we now show that if \(f\) satisfies a Lipschitz condition on a strip \(S^\mathbb{C} := \{(t, p) \in \mathbb{T}^\mathbb{C} \times \mathbb{R} : t \in [t_0, t_0 + a]^\mathbb{T}, |p| < \infty\}\), rather than on the rectangle \(R^\mathbb{C}\), then solutions will exist on the entire interval \([t_0, t_0 + a]^\mathbb{T}\).

Theorem 4.13 Let \(f\) be Hilger continuous on the strip \(S^\mathbb{C}\) and suppose there exists a constant \(L > 0\) such that
\[ |f(t, p) - f(t, q)| \leq L|p - q|, \quad \text{for all } (t, p), (t, q) \in S^\mathbb{C}. \]

Then the successive approximations \(\phi_k\) exist as continuous functions on the entire interval \(J := [t_0, t_0 + a]^\mathbb{T}\) and converge there uniformly to the unique solution of (2), (3).

Proof: The proof follows similar lines to that of the proof of Theorem 4.4 and so is only sketched.

An induction argument establishes that each \(\phi_k\) exists as a continuous function on \(J\).

A constant \(M_1 > 0\) exists such that
\[ |f(t, x_0)| \leq M_1, \quad \text{for all } t \in J^\mathbb{C}. \]
The uniform convergence of \(\phi_k\) to some function \(\phi\) on \(I\) can now be established via the estimate
\[ |\phi_i(t) - \phi_{i-1}(t)| \leq \frac{M_1 L^i(t - t_0)^i}{i!}, \quad i = 1, 2, \ldots \]  
(30)
for all \( t \in J \); and then employing Weierstrass’ test. The estimate (30) is shown via induction, starting with

\[
|\phi_1(t) - \phi_0(t)| = \left| \int_{t_0}^{t} f(s, x_0) \Delta s \right| \leq M_1(t - t_0)
\]

for all \( t \in J \).

We then see that for each integer \( k \geq 0 \) and all \( t \in J \) we have

\[
|\phi_k(t) - x_0| \leq \sum_{i=1}^{k} |\phi_i(t) - \phi_{i-1}(t)| \leq M_1 \frac{\infty}{L} \sum_{i=1}^{\infty} \frac{L^i(t - t_0)^i}{i!} = M_1 \frac{L}{e^{La} - 1} =: b.
\]

If we now define the rectangle \( R \) as in (9) with the above definition of \( b \) then we see that each of the graphs of \((t, \phi_k(t))\) and \((t, \phi(t))\) stay within \( R \) for all \( t \in J \).

If \( M \) is the bound of \( f \) on \( \mathbb{R}^\kappa \) then the continuity of \( \phi \) on \( J \) is proved in a similar way as in the proof of Theorem 4.4 and so is omitted.

The remainder of the proof where convergence of \( \int_{t_0}^{t} f(s, \phi_k(s)) \Delta s \) and uniqueness of solutions established via Lemma 3.2 also follows the arguments of the proof of Theorem 4.4 and so is also omitted.

As a corollary to Theorem 4.13 we have the following well-known result of Hilger [17, Theorem 5.7] that guarantees the existence of a unique solution to the dynamic IVP (2), (3) on the half-line \([t_0, \infty)\).

**Corollary 4.14** Let \( f \) be Hilger continuous on the half-plane

\[
P := \{(t, p) \in \mathbb{T} \times \mathbb{R} : t \in [t_0, \infty), |p| < \infty\}.
\]

Assume that \( f \) satisfies a Lipschitz condition

\[
|f(t, p) - f(t, q)| \leq L_{t_0, a}|p - q|
\]

on each strip

\[
S_{t_0, a}^\kappa := \{(t, p) \in \mathbb{T}^\kappa \times \mathbb{R} : t \in [t_0, t_0 + a]|_{\mathbb{T}}, |p| < \infty\},
\]

where \( L_{t_0, a} \) is a constant that may depend on \( t_0 \) and \( a \). Then, the dynamic IVP (2), (3) has a unique solution that exists on the whole half-line \([t_0, \infty)\).

**Proof:** The proof involves showing that the conditions of Theorem 4.13 hold on every strip of \( S_{t_0, a}^\kappa \) and is omitted for brevity. \( \square \)

In the following theorem we introduce a linear growth condition on \( f \) and use a Lipschitz condition to ensure the nonlocal existence and uniqueness of solutions to (2), (3).
Theorem 4.15 Let \( f \) be Hilger continuous on the rectangle \( R^\kappa \) in (8). Let \( l : [t_0, t_0 + a] \to [0, \infty) \) be a continuous, nondecreasing function such that

\[
|f(t, p)| \leq l(t)(1 + |p|), \quad \text{for all } (t, p) \in R^\kappa; \tag{31}
\]

\[
|f(t, p) - f(t, q)| \leq l(t)|p - q|, \quad \text{for all } (t, p), (t, q) \in R^\kappa; \tag{32}
\]

\[
(1 + |x_0|) \left[ \exp \left( \int_{t_0}^{t_0 + a} l(s) \, ds \right) - 1 \right] \leq b. \tag{33}
\]

Then the successive approximations converge uniformly on \( J \) to the unique solution of the dynamic IVP (2), (3).

Proof: For clarity, we divide the proof into smaller sections.

1. Continuity and boundedness of \( \phi_k \):

   The continuity of each \( \phi_k \) on \( J := [t_0, t_0 + a] \) follows by induction as in the proof of Theorem 4.4 and so is omitted.

   We show that each \( \phi_k \) satisfies

   \[
   |\phi_k(t) - x_0| \leq (1 + |x_0|) \left[ \exp \left( \int_{t_0}^{t} l(s) \, ds \right) - 1 \right] \tag{34}
   \]

   for all \( t \in J \) and so \((t, \phi_k(t)) \in R \) for all \( t \in J \). We use induction. For \( k = 0 \) see that (34) trivially holds. Now assume (34) holds for some \( k = n \geq 0 \), so that, in particular,

   \[
   1 + |\phi_n(t)| \leq (1 + |x_0|) \left[ \exp \left( \int_{t_0}^{t} l(s) \, ds \right) \right], \quad \text{for all } t \in J.
   \]

   If we define \( L(t) := \int_{t_0}^{t} l(s) \, ds \) for all \( t \in J \) then see that for all \( t \in J \) we have

   \[
   |\phi_{n+1}(t) - x_0| = \left| \int_{t_0}^{t} f(s, \phi_n(s)) \, \Delta s \right|
   \leq \int_{t_0}^{t} l(s)(1 + |\phi_n(s)|) \, \Delta s
   \leq \int_{t_0}^{t} l(s)(1 + |x_0|)e^{L(s)} \, \Delta s
   \leq (1 + |x_0|) \int_{t_0}^{t} l(s)e^{L(s)} \, ds
   = (1 + |x_0|)(e^{L(t)} - 1)
   \leq b,
   \]

   where we have used: (31), (33); the fact that \( l \) is nondecreasing; and Lemma 4.3. Hence (34) is true for \( k = n + 1 \) and so holds in general by induction.
2. Uniform convergence of $\phi_k$ on $J$: It is sufficient to show that for each $t \in J$ we have the estimate

$$|\phi_i(t) - \phi_{i-1}(t)| \leq (1 + |x_0|)\frac{[L(t)]^i}{i!}, \quad i = 1, 2, \ldots$$

(35)

To do this, we use induction. For $i = 1$ see that

$$|\phi_1(t) - \phi_0(t)| \leq \int_{t_0}^t |f(s, x_0)| \Delta s \leq (1 + |x_0|)L(t),$$

where we have used (31). Thus, (35) holds for $i = 1$. Now assume that (35) holds for some $i = n \geq 1$. See that

$$|\phi_{n+1}(t) - \phi_n(t)| \leq \int_{t_0}^t |f(s, \phi_n(s)) - f(s, \phi_{n-1}(s))| \Delta s$$

$$\leq \int_{t_0}^t l(s)|\phi_n(s) - \phi_{n-1}(s)| \Delta s$$

$$\leq (1 + |x_0|)\int_{t_0}^t l(s)\frac{[L(s)]^n}{n!} \Delta s$$

$$\leq (1 + |x_0|)\int_{t_0}^t l(s)\frac{[L(s)]^n}{n!} ds$$

$$= (1 + |x_0|)\frac{[L(t)]^{n+1}}{(n+1)!}$$

where we have used (31), (32), the fact that $l$ is nondecreasing, and Lemma 4.3. Hence (35) is true for $i = n + 1$ and so holds in general by induction.

Thus, for all $t \in J$ we have

$$|\phi_i(t) - \phi_{i-1}(t)| \leq (1 + |x_0|)\frac{[L(t_0 + a)]^i}{i!}, \quad i = 1, 2, \ldots,$$

and since

$$\sum_{i=1}^{\infty} \frac{[L(t_0 + a)]^i}{i!}$$

converges (to $e^{L(t_0 + a)} - 1$), Weierstrass’ test applies and we conclude that $\phi_k$ converges uniformly on $J$ to some function $\phi$.

3. The limit function $\phi$ is the unique solution:

The remainder of the proof also follows the arguments of the proof of Theorem 4.4 and so is also omitted.

\[\square\]
5 An interesting example

The results of the preceding section raise the question on whether the Hilger continuity of $f$ alone is sufficient to guarantee the convergence of the sequence (or a subsequence) of successive approximations. We now construct an example that answers this question.

Let $\theta : [0, 1]_T \rightarrow [0, \infty)$ be a delta differentiable function that satisfies $\theta(0) = 0$; $\theta(t) > 0$ for all $t \in (0, 1]_T$; $\theta^\Delta(t) > 0$ for all $t \in [0, 1]_T$; and $\theta^\Delta \in C_{rd}$.

Consider the dynamic IVP

\[
x^\Delta = f(t, x), \quad t \in [0, 1]_T; \quad x(0) = 0;
\]

with $f$ defined by

\[
f(t, p) := \begin{cases} 
0, & \text{for all } t = 0, -\infty < p < \infty; \\
\theta^\Delta(t), & \text{for all } t \in (0, 1]_T, p \leq 0; \\
\theta^\Delta(t) - \frac{\theta^\Delta(t)}{\theta(t)} p, & \text{for all } t \in (0, 1]_T, 0 < p \leq \theta(t); \\
0, & \text{for all } t \in (0, 1]_T, p > \theta(t). 
\end{cases}
\]

From (38) it can be seen that $f(t, p)$ is Hilger–continuous for all $(t, p) \in [0, 1]_T \times \mathbb{R}$.

Defining our sequence of successive approximations as in (6), (7) we see that for all $t \in [0, 1]_T$

\[
\phi_1(t) = \int_0^t f(s, \phi_0(s)) \Delta s \\
= \int_0^t f(s, 0) \Delta s \\
= \int_0^t \theta^\Delta(s) \Delta s \\
= \theta(t).
\]

For all $t \in [0, 1]_T$ the second iteration will be

\[
\phi_2(t) = \int_0^t f(s, \phi_1(s)) \Delta s \\
= \int_0^t f(s, \theta(s)) \Delta s \\
= \int_0^t \theta^\Delta(s) - \frac{\theta^\Delta(s)}{\theta(s)} \theta(s) \Delta s \\
= 0.
\]
Similarly, \( \phi_3(t) = \theta(t) \) and \( \phi_4(t) = 0 \) for all \( t \in [0,1]_T \). In this way, for each integer \( k \geq 0 \), we obtain
\[
\phi_{2k}(t) = 0 \quad \text{and} \quad \phi_{2k+1}(t) = \theta(t), \quad \text{for all} \quad t \in [0,1]_T.
\]
See that \( \phi_{2k} \to 0 \) and \( \phi_{2k+1} \to \theta \) on \( [0,1]_T \). As each of these subsequences converge to different limits, \( \phi_k \) cannot converge to a limit on \( [0,1]_T \). Furthermore, neither of the limits of the above subsequences are actually solutions to (36), (37) as may be seen by verifying
\[
f(t,0) \neq 0, \quad f(t,\theta(t)) \neq \theta(t), \quad \forall t \in (0,1) \kappa_T.
\]
In addition, it can be shown that a solution to (36), (37) is \( x(t) = \theta(t)/2 \) for all \( t \in [0,1]_T \). This solution cannot be obtained by the above scheme of successive approximations.

The above example is a generalisation to the time scales setting of a very famous example constructed for the case of ordinary differential equations by Müller [24] and Reid [29, pp.51–52].

## 6 More on approximation of solutions

In this section we focus on the approximation of solutions to the dynamic IVP (2), (3). We establish some conditions under which we can estimate maximum errors between true solutions and approximative solutions; and we also discover when solutions of certain approximative problems will converge to solutions of the dynamic IVP of interest.

**Definition 6.1** We call a continuous function \( \varphi : J \subset T \to \mathbb{R} \) an \( \varepsilon \)-approximate solution to (2) on \( J^e \) if:

1. the graph of \( (t, \varphi(t)) \) lies in \( J \times D \) for all \( t \in J \);
2. the function \( \varphi^\Delta \) is rd-continuous on \( J^e \);
3. there exists an \( \varepsilon \geq 0 \) such that \( |\varphi^\Delta(t) - f(t, \varphi(t))| \leq \varepsilon \) for all \( t \in J^e \).

**Remark 6.2** The definition of an \( \varepsilon \)-approximate solution can be relaxed. For example, \( \varphi^\Delta \) could have simple discontinuities at a finite number of points \( S \) in \( J^e \); and part 3 above would be then relaxed to hold for all \( t \in J^e - S \). However, the simpler definition will suffice for our requirements.

The following result gives an estimate on the difference of two \( \varepsilon \)-approximate solutions to (2).

**Theorem 6.3** Let \( J \) be an interval of \( T \) bounded below such that \( t_0 \) is the left-hand end point of \( J \). Let \( f : J^e \times D \subseteq T^e \times \mathbb{R} \to \mathbb{R} \) be H"{u}ller continuous and let \( L > 0 \) be a constant such that
\[
|f(t,p) - f(t,q)| \leq L|p - q|, \quad \text{for all} \quad (t,p),(t,q) \in J^e \times D.
\]
Let \( \varphi_1 \) and \( \varphi_2 \) be, respectively, \( \varepsilon_1 \) and \( \varepsilon_2 \) approximate solutions of (2) on \( J^e \). If \( \delta \geq 0 \) is a constant such that
\[
|\varphi_1(t_0) - \varphi_2(t_0)| \leq \delta,
\]
then \( \varphi_1 \) and \( \varphi_2 \) satisfy
\[
|\varphi_1(t) - \varphi_2(t)| \leq \delta e(t, t_0) + \frac{\epsilon_1 + \epsilon_2}{L} (e(t, t_0) - 1), \quad \text{for all } t \in J. \tag{40}
\]

**Proof:** Since each \( \varphi_i \) is, respectively, an \( \epsilon_i \)-approximate solution to (2), we have
\[
|\varphi_i^\Delta(t) - f(t, \varphi_i(t))| \leq \epsilon_i, \quad \text{for all } t \in J^\infty \tag{41}
\]
for \( i = 1, 2 \). Thus, integrating both sides of (41) from \( t_0 \) to \( t \) we obtain
\[
\epsilon_i(t - t_0) \geq \left| \varphi_i(t) - \varphi_i(t_0) - \int_{t_0}^{t} f(x, \varphi_i(s)) \Delta s \right|, \quad \text{for all } t \in J. \tag{42}
\]
Adding together the two inequalities contained in (42) and using the inequality \( |\alpha - \beta| \leq |\alpha| + |\beta| \) we then obtain
\[
(\varphi_1(t) - \varphi_2(t)) - (\varphi_1(t_0) - \varphi_2(t_0)) - \int_{t_0}^{t} f(s, \varphi_1(s)) - f(s, \varphi_2(s)) \Delta s \leq \epsilon(t - t_0) \tag{43}
\]
for all \( t \in J \); and where \( \epsilon := \epsilon_1 + \epsilon_2 \).

Now let
\[
r(t) := |\varphi_1(t) - \varphi_2(t)|, \quad \text{for all } t \in J.
\]
Substitution into (43) coupled with a rearrangement and use of the inequality \( |\alpha - \beta| \leq |\alpha| + |\beta| \)
then gives for all \( t \in J \)
\[
r(t) \leq r(t_0) + \int_{t_0}^{t} |f(s, \varphi_1(s)) - f(s, \varphi_2(s))| \Delta s + \epsilon(t - t_0)
\leq \delta + \int_{t_0}^{t} L r(s) \Delta s + \epsilon(t - t_0),
\]
where we have used (39). An application of Grönwall’s inequality then yields
\[
r(t) := |\varphi_1(t) - \varphi_2(t)|
\leq \delta + \epsilon(t - t_0) + L e(t_0, t_0) \int_{t_0}^{t} \left[ \delta + \epsilon(s - t_0) \right] \frac{1}{e(\sigma(s), t_0)} \Delta s
\]
for all \( t \in J \), which may be further simplified to the desired inequality (40) via integration by parts [6, Theorem 1.77 (v)]. \( \square \)

**Remark 6.4** If \( \varphi_1 \) is an actual solution to (2) then we have \( \epsilon_1 = 0 \) in (40). Thus, under the conditions of Theorem 6.3 we see that \( \varphi_2 \to \varphi_1 \) as \( \epsilon_2 \to 0 \) and \( \delta \to 0 \).

**Remark 6.5** Inequality (40) is the best inequality possible in the sense that equality can be attained for nontrivial \( \varphi_1 \) and \( \varphi_2 \).
**Proof:** For example, consider the linear equation

$$x^\Delta = Lx, \quad t \in J^c;$$

(44)

where $L > 0$ is a constant. Let $\varphi_1$ and $\varphi_2$ be respective solutions to

$$x^\Delta = Lx - \varepsilon_1, \quad x^\Delta = Lx + \varepsilon_2;$$

for some constants $\varepsilon_1$, $\varepsilon_2$. Further, let $\varphi_1$, $\varphi_2$ satisfy $\varphi_1(t_0) = b_1$ and $\varphi_2(t_0) = b_2$ where $b_1$ and $b_2$ are constants such that $b_1 \leq b_2$. Then from the theory of linear dynamic equations [6, Chap.3] we see for all $t \in J$

$$\varphi_1(t) = \frac{\varepsilon_1}{L} + \left( b_1 - \frac{\varepsilon_1}{L} \right) e_L(t, t_0);$$

$$\varphi_2(t) = -\frac{\varepsilon_2}{L} + \left( b_2 + \frac{\varepsilon_2}{L} \right) e_L(t, t_0).$$

Thus, for all $t \in J$ we have

$$|\varphi_1(t) - \varphi_2(t)| = \left| \frac{\varepsilon_1 + \varepsilon_2}{L} (1 - e_L(t, t_0)) + (b_1 - b_2)e_L(t, t_0) \right|$$

and so equality holds in (40).

Let the rectangle $R$ be defined in (9) and let $f, g : R^c \to \mathbb{R}$. Consider the two dynamic IVPs

$$x^\Delta = f(t, x), \quad x(t_0) = \alpha; \quad (45)$$

$$x^\Delta = g(t, x), \quad x(t_0) = \beta; \quad (46)$$

where $\alpha$ and $\beta$ are constants such that $(t_0, \alpha)$ and $(t_0, \beta)$ both lie in $R$. We now provide a result that relates the solutions of (45) and (46) in the sense that if $f$ is close to $g$ and if $\alpha$ is close to $\beta$ then the two solutions are close together also.

**Theorem 6.6** Let $f$ and $g$ both be Hilger continuous on the rectangle $R^c$. Let there exist constants $L > 0$, $\varepsilon \geq 0$ and $\delta \geq 0$ such that

$$|f(t, p) - f(t, q)| \leq L|p - q|, \quad \text{for all } (t, p), (t, q) \in R^c;$$

$$|f(t, p) - g(t, p)| \leq \varepsilon, \quad \text{for all } (t, p) \in R^c;$$

$$|\alpha - \beta| \leq \delta.$$

Let $x_1$ and $x_2$ be, respectively, solutions to (45) and (46) such that each $(t, x_i(t)) \in R$ for all $t \in J$. Then

$$|x_1(t) - x_2(t)| \leq \delta e_L(t, t_0) + \varepsilon e_L(t, t_0) - 1), \quad \text{for all } t \in J. \quad (47)$$
Proof: For each \( t \in J \), consider
\[
\begin{align*}
  r(t) & := |x_1(t) - x_2(t)| \\
  & = |\alpha - \beta + \int_{t_0}^{t} f(s, x_1(s)) - g(s, x_2(s)) \Delta s| \\
  & \leq \delta + \int_{t_0}^{t} |f(s, x_1(s)) - f(s, x_2(s))| \Delta s + \int_{t_0}^{t} |f(s, x_2(s)) - g(s, x_2(s))| \Delta s \\
  & \leq \delta + \int_{t_0}^{t} L r(s) \Delta s + \varepsilon(t - t_0).
\end{align*}
\]

We can now apply Grönwall’s inequality above and integrate by parts as in the proof of Theorem 6.3 to obtain (47).

Now consider (45) together with the sequence of problems
\[
x^k = g_k(t, x), \quad x(t_0) = \alpha_k, \quad k = 1, 2, \ldots
\]
where \( g_k \) and \( \alpha_k \) are sequences with each \( g_k \) being Hilger continuous on \( \mathbb{R}_\kappa \) and each \((t_0, \alpha_k)\) being in \( \mathbb{R} \). As a corollary to Theorem 6.6 we now establish a result that ensures solutions to (48) will converge to solutions to (45).

Corollary 6.7 Let \( f \) and each \( g_k \) be Hilger continuous on \( \mathbb{R}_\kappa \). Let \( f \) and each \( g_k \) satisfy
\[
\begin{align*}
  |f(t, p) - f(t, q)| & \leq L|p - q|, \quad \text{for all } (t, p), (t, q) \in \mathbb{R}_\kappa; \\
  |f(t, p) - g_k(t, p)| & \leq \varepsilon_k, \quad \text{for all } (t, p) \in \mathbb{R}_\kappa; \\
  |\alpha - \alpha_k| & \leq \delta_k,
\end{align*}
\]
where \( L > 0 \) is a constant with \( \varepsilon_k \to 0 \) and \( \delta_k \to 0 \) as \( k \to \infty \). If \( x_k \) is a solution of (48) and \( x \) is a solution to (45) on \( J \) such that each \((t, x(t)), (t, x_k(t))\) \( \in \mathbb{R} \) for all \( t \in J \) then \( x_k \to x \) on \( J \).

Proof: For each \( k \), the conditions of Theorem 6.6 hold with the result following from an application of (47).

7 Comparison results for the exponential function on \( \mathbb{T} \)

In this section we apply the method of successive approximation to obtain a comparison result between the general exponential function on a time scale and the classical exponential function.

Theorem 7.1 Let \( m \in \mathbb{R} \) be a constant. Then
\[
e_m(t, t_0) \leq e^{|m|(t - t_0)}, \quad \text{for all } t \geq t_0.
\]
Proof: Consider the linear dynamic IVP
\[ x^\Delta = mx, \quad t \geq t_0; \quad x(t_0) = 1. \]

Now consider the sequence \( \phi_k \) defined by
\[
\phi_0(t) \equiv 1; \\
\phi_{k+1}(t) := 1 + \int_{t_0}^t m\phi_k(s) \Delta s, \quad k = 1, 2, \ldots.
\]

Corollary 4.14 may be applied to the above linear IVP and so we know that \( \phi_k \) will converge to a function \( \phi \) which will be a solution to the original IVP. Since \( m \in \mathcal{R} \), the solution is given by \( \phi(t) := e_m(t, t_0), \ t \geq t_0. \)

We show that
\[
\phi_k(t) \leq e^{\left|m(t-t_0)\right|}, \quad \text{for all } t \geq t_0.
\]

To establish (55) we employ proof by induction. See that
\[
\phi_0(t) := 1 \leq e^{\left|m(t-t_0)\right|}, \quad \text{for all } t \geq t_0,
\]
and so (52) holds for \( k = 0. \)

Now assume (55) holds for some \( k = n \geq 0. \) See, for all \( t \geq t_0 \)
\[
\phi_{n+1}(t) := 1 + \int_{t_0}^t m\phi_n(s) \Delta s \\
\leq 1 + |m| \int_{t_0}^t e^{\left|m(s-t_0)\right|} \Delta s \\
\leq 1 + |m| \int_{t_0}^t e^{\left|m(s-t_0)\right|} ds \\
= e^{\left|m(t-t_0)\right|},
\]
and thus, by induction, (55) holds for all \( k. \)

Taking limits as \( k \to \infty \) in (55) we then obtain
\[
\phi(t) := e_m(t, t_0) \leq e^{\left|m(t-t_0)\right|}, \quad \text{for all } t \geq t_0.
\]

Remark 7.2 There are many other comparison results possible as a result of the above method. For example, applying the method to
\[ x^\Delta = tx, \quad t \geq t_0; \quad x(t_0) = 1, \]
with \( p(t) := t \) and \( p \in \mathcal{R} \) gives
\[ e_p(t, t_0) \leq e^{(t-t_0)^2/2}, \quad t \geq t_0 \geq 0. \]

For other methods concerned with comparison results for the general exponential function on a time scale see [14, Sec.3] and [28, Sec.2].
8 Generalisations

In this section we briefly illustrate some generalisations of the methods and results from previous sections.

In our “standard” definition of successive approximations (6), (7) see that we defined the starting approximation $\phi_0$ as the constant function $x_0$ on a subset of $J$. We now suitably relax this restriction.

We define a sequence of functions $\phi_0, \phi_1, \phi_2, \ldots$ each defined on some subset of $J := [t_0, t_0 + a] T$ in the following way:

$$\phi_0 \text{ is chosen as a continuous function on a subset of } J;$$

$$\phi_{k+1}(t) := x_0 + \int_{t_0}^{t} f(s, \phi_k(s)) \Delta s, \quad k = 1, 2, \ldots$$

We now present a Picard–Lindelöf result for the above scheme.

**Theorem 8.1** Let $f$ be Hilger–continuous on the rectangle $R^\kappa$ and let $\alpha$ be defined in (11). Suppose there exists a constant $L > 0$ such that $f$ satisfies

$$|f(t, p) - f(t, q)| \leq L|p - q|, \quad \text{for all } (t, p), (t, q) \in R^\kappa.$$  

Then the successive approximations $\phi_k$ defined in (56), (57) converge uniformly on $I := [t_0, t_0 + \alpha] T$ to the unique solution of (2), (3) provided $(t, \phi_0(t)) \in R$ for all $t \in J$.

**Proof:** The proof is omitted as it is very similar to that of the proof of Theorem 4.4. The only distinction involves the estimate

$$|\phi_i(t) - \phi_{i-1}(t)| \leq N \frac{L(t - t_0)^{i-1}}{(i-1)!}, \quad i = 1, 2, \ldots$$

for all $t \in I$, where $N$ is a constant such that

$$\max_{t \in I} |\phi_i(t) - \phi_0(t)| \leq N.$$

The choice regarding $\phi_0$ in a successive approximation scheme is interesting for two reasons. Firstly, varying the choices of $\phi_0$ can lead to improved existence results. Secondly, in the case of multiple solutions to a dynamic IVP, different choices of $\phi_0$ can lead to $\phi_k$ converging to different solutions, see [5, p.120] for a discussion of the special case $T = R$.

We now illustrate the first point raised above with an extension of Theorem 4.10.

**Theorem 8.2** Let $f$ be Hilger continuous on the rectangle $R^\kappa$ and let $\alpha$ be defined in (11). If there exists a continuous function $\phi_0$ defined on $J$ such that $(t, \phi_0(t)) \in R$ for all $t \in J$ and

$$f(t, p) \leq f(t, q), \quad \text{for all } p \leq q, \text{ and all } t \in J^\kappa;$$

$$f(t, \phi_0(t)) \leq 0, \quad \text{for all } t \in J^\kappa;$$

then the successive approximations $\phi_k$ defined in (56), (57) converge uniformly on the interval $I := [t_0, t_0 + \alpha] T$ to a solution of the dynamic IVP (2), (3).
Proof: The proof follows the same ideas as that of the proof of Theorem 8.2 and so is omitted. □

Remark 8.3 See that condition (60) is less restrictive than (27). Also note that we may farther extend Theorem 8.2 by replacing (60) with

\[ \int_I f(t, \phi_0(t)) \Delta t \leq 0, \]

(see Remark 4.11).

Remark 8.4 We may now use the above improved scheme (56), (57) when returning to the example presented in Section 5. If we begin our sequence of approximations with \( \phi_0(t) = \theta(t)/2 \) then the solution to our problem is easily obtained as the limit of the \( \phi_k \).

Remark 8.5 For simplicity we have restricted our attention in this paper to scalar–valued \( f \) in (2). The method of successive approximations can be extended to the case when \( f \) is vector–valued. This, in turn, allows the application of the method of successive approximations to higher–order IVPs.

Remark 8.6 All of the results contained in this work only concen the existence and behaviour of solutions “to the right” of the initial point \( t_0 \). Many of the ideas contained herein can be extended so as to also apply “to the left” of \( t_0 \), in addition to the right. The key assumption on \( f \) to add to all of the theorems would be “f is regressive” in the sense of [6, p.322].

Remark 8.7 If our time scale \( \mathbb{T} \) contains only isolated points, so that the interval \([t_0, t_0+\alpha_0]_{\mathbb{T}}\) contains a finite number of points, then (6), (7) may be solved recursively to determine the unique solution.

Remark 8.8 Note that our results contained herein apply to time scales that feature both right–scattered and right–dense points.

9 Conclusion and Future Directions

This article has established the method of “successive approximations” for initial value problems associated with dynamic equations on time scales. We believe that it would be very interesting to investigate other types of dynamic equations on time scales, such as boundary value problems [10, 11, 15, 16, 25, 31], to see if insight can be gained via the method of successive approximations.

References


