On the Existence of Solutions to Boundary Value Problems on Time Scales

Christopher C. Tisdell\(^1\) and H. B. Thompson\(^2\)

\(^1\)School of Mathematics
The University of New South Wales
Sydney NSW 2052, AUSTRALIA
\(^2\)Department of Mathematics
The University of Queensland
St Lucia QLD 4072, AUSTRALIA

Abstract. This work formulates existence theorems for solutions to two-point boundary value problems on time scales. The methods used include maximum principles, a priori bounds and topological degree theory.

Keywords. time scale, measure chain, boundary value problem, topological degree, dynamic equation.

AMS (MOS) subject classification: 39A12.

1 Introduction

Motivated by the desire to unify the theory of continuous and discrete calculus, Stefan Hilger [11] introduced the theory of time scales in 1990. Hilger defined the “generalized derivative” \( y^\Delta(t) \), where the domain of the function is a so-called “time scale” (which is an arbitrary closed subset of \( \mathbb{R} \)). By choosing the time scale to be, say \( \mathbb{R} \), then the generalized derivative is just the usual derivative from calculus, ie \( y^\Delta(t) = y'(t) \). By choosing the time scale to be, say \( \mathbb{Z} \), then the generalized derivative is just the usual forward difference, ie \( y^\Delta(t) = \Delta y(t) \). There are many more time scales than just these two cases.

This paper considers the existence of solutions to the second-order dynamic equation

\[ y^{\Delta\Delta}(t) = f(t, y(\sigma(t))), \quad t \in [a, b], \]  

subject to the separated boundary conditions

\[ g((y(a), y(\sigma^2(b))); (y^\Delta(a), y^\Delta(\sigma(b)))) = (0, 0), \]  

where \( f : [a, b] \times \mathbb{R} \to \mathbb{R} \) and \( t \) is from a so-called “time scale”. Together, equations (1) and (2) are known as a boundary value problem (BVP) on time scales. Once again, by choosing the time scale to be, say \( \mathbb{R} \), then (1) will
be a second-order, ordinary differential equation. By choosing the time scale to be, say $\mathbb{Z}$, then (1) will be a second-order difference equation. There are many more BVPs on time scales than just these two cases.

To understand the notation used above and the idea of time scales some preliminary definitions are needed.

**Definition** A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$.

Since a time scale may or may not be connected, the concept of jump operators is needed to overcome this difficulty.

**Definition** Define the forward (backward) jump operator $\sigma(t)$ at $t < \sup \mathbb{T}$ ($\rho(t)$ at $t > \inf \mathbb{T}$) by

$$\sigma(t) = \inf \{ \tau > t : \tau \in \mathbb{T} \}, \quad (\rho(t) = \sup \{ \tau < t : \tau \in \mathbb{T} \},)$$

for all $t \in \mathbb{T}$.

For simplicity and clarity denote $\sigma^2(t) = \sigma(\sigma(t))$ and $y''(t) = y(\sigma(t))$.

If $\mathbb{T} = \mathbb{R}$ then $\sigma(t) = t = \rho(t)$. If $\mathbb{T} = \mathbb{Z}$ then $\sigma(t) = t + 1$ and $\rho(t) = t - 1$.

Throughout this work the assumption is made that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. Also assume throughout that $a < b$ are points in $\mathbb{T}$ with $[a, b] = \{ t \in \mathbb{T} : a \leq t \leq b \}$.

The jump operators $\sigma$ and $\rho$ allow the classification of points in a time scale in the following way: If $\sigma(t) > t$ then call the point $t$ right-scattered; while if $\rho(t) < t$ then say $t$ is left-scattered. If $\sigma(t) = t$ then call the point $t$ right-dense; while if $\rho(t) = t$ then say $t$ is left-dense.

If $\mathbb{T}$ has a left-scattered maximum $m$ then define $\mathbb{T}^k = \mathbb{T} - \{ m \}$. Otherwise $\mathbb{T}^k = \mathbb{T}$.

**Definition** Fix $t \in \mathbb{T}$ and let $y : \mathbb{T} \to \mathbb{R}$. Define $y^\Delta(t)$ to be the number (if it exists) with the property that given $\epsilon > 0$ there is a neighbourhood $U$ of $t$ with

$$\left| y(\sigma(t)) - y(s) - y^\Delta(t)[\sigma(t) - s] \right| < \epsilon|\sigma(t) - s|, \quad \text{for all } s \in U.$$

Call $y^\Delta(t)$ the derivative of $y(t)$. Define the second derivative by $y^{\Delta \Delta} = (y^\Delta)^\Delta$.

The following theorem is due to Hilger [11].

**Theorem 1** Assume that $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^k$.

(i) If $f$ is differentiable at $t$ then $f$ is continuous at $t$.

(ii) If $f$ is continuous at $t$ and $t$ is right-scattered then $f$ is differentiable at $t$ with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

(iii) If $f$ is differentiable and $t$ is right-dense then

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) If $f$ is differentiable at $t$ then $f(\sigma(t)) = f(t) + (\sigma(t) - t)f^\Delta(t)$. 

Definition If $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^h$ then define the integral by
\[ \int_a^t f(s)\Delta s = F(t) - F(a). \]

Definition Assume $f : \mathbb{T} \to \mathbb{R}$. Define and denote $f \in C_{rd}(\mathbb{T}; \mathbb{R})$ as right-dense continuous if at all $t \in \mathbb{T}$ then:
\[ \lim_{s \to t^+} f(s) = f(t), \]
at every right-dense point $t \in \mathbb{T}$,
\[ \lim_{s \to t^-} f(s) \text{ exists and is finite}, \]
at every left-dense point $t \in \mathbb{T}$.

The forward jump operator $\sigma(t) \in C_{rd}(\mathbb{T})$.

Definition Define $C^2_{rd}([a, \sigma^2(b)])$ to be the set of all functions $y : \mathbb{T} \to \mathbb{R}$ such that
\[ C^2_{rd}([a, \sigma^2(b)]) = \{ y : y \in C([a, \sigma^2(b)]) \text{ and } y^{\Delta\Delta} \in C_{rd}([a, b]) \}. \]

A solution to (1) is a function $y \in C^2_{rd}([a, \sigma^2(b)])$ which satisfies (1) for each $t \in [a, b]$.

The BVPs treated in this paper include a very wide range of boundary conditions, including nonlinear variations. The motivation for the research in this paper came from the works [1] - [4] and [6] - [8].

The monographs [5] and [12] also provide an excellent introduction to the theory of time scales.

2 A Priori Bounds on Solutions

The ideas and methods used in this paper rely on “degree theory” (sometimes called topological degree theory). Suppose that we want to prove that the equation $z(x) = p$ has a solution $x$. Here $z \in C(D; X)$, $D$ is an open and bounded subset of the normed space $X$ and $p \in X$.

The triple $d(z, D, p)$, (provided it is defined,) gives us an integer referred to as “the degree of $z$ at $p$ relative to $D$”. In particular, the following two theorems are the essence of degree theory and their proofs can be found in Lloyd [14].

Theorem 2 Let $z \in C(D; X)$ and $D \subset X$ be finite-dimensional with $D$ open and bounded. If $z(x) \neq p$ for all $x \in \partial D$ and if $d(z, D, p) \neq 0$ then the equation $z(x) = p$ has at least one solution $x \in D$.

Theorem 3 Let $z \in K_1(D; X)$ where $K_1$ is the set of compact perturbations of the identity and $D \subset X$ is infinite-dimensional with $D$ open and bounded. If $z(x) \neq p$ for all $x \in \partial D$ and if $d(z, D, p) \neq 0$ then the equation $z(x) = p$ has at least one solution $x \in D$.

The question now naturally arises on how to calculate the integer $d(z, D, p)$. There are many different ways of doing this and we refer the reader to Lloyd.
[14] for a detailed discussion. A simple fact that we now state is for the case \( z(x) = I(x) = x \).

\[
d(I, D, p) = \begin{cases} 
  1, & \text{if } p \in D, \\
  0, & \text{if } p \notin D.
\end{cases}
\]

This will be particularly useful when dealing with “homotopy methods” in the sections ahead.

One way of ensuring that solutions to the equation \( z(x) = p \) satisfy \( x \notin \partial D \) is the method of a priori bounds. Here all solutions to \( z(x) = p \) are shown to be bounded a priori such that \( x \in D \) and therefore there are no solutions with \( x \in \partial D \). In the remainder of this section, conditions on \( f \) and on the boundary conditions are formulated under which these bounds are guaranteed for solutions \( y \) to the BVP (1), (2).

The following maximum principle shall be very useful throughout the rest of the paper and can be found in Gnana Bhaskar [8].

**Lemma 1** If a function \( r : \mathbb{T} \to \mathbb{R} \) has a local maximum at a point \( c \in (a, \sigma^2(b)) \) then \( r^\Delta(\rho(c)) \leq 0 \) provided \( c \) is not simultaneously left-dense and right-scattered and that \( r^\Delta(\rho(c)) \) exists.

**Lemma 2** Let \( \alpha, \beta \in C_{rd}^2([a, \sigma^2(b)]) \) satisfy \( \alpha \leq \beta \),

\[
\alpha^\Delta > f(t, y^\sigma), \quad \text{for } y^\sigma < \alpha^\sigma \text{ and } t \in [a, b],
\]
\[
\beta^\Delta < f(t, y^\beta), \quad \text{for } y^\beta > \beta^\sigma \text{ and } t \in [a, b].
\]

If \( y \) is a solution to (1) with \( \alpha(a) \leq y(a) \leq \beta(a) \) and \( \alpha(\sigma^2(b)) \leq y(\sigma^2(b)) \leq \beta(\sigma^2(b)) \) then \( \alpha \leq y \leq \beta \) for \( t \in [a, \sigma^2(b)] \).

**Proof** Let \( y \) be a solution to (1) and assume that the conclusion of the lemma is false. In particular, suppose that \( y(c) > \beta(c) \) for some \( c \in [a, \sigma^2(b)] \).

Now the function \( r(t) = y(t) - \beta(t) \) is continuous and must have a positive maximum, \( M \), in \([a, \sigma^2(b)]\). By hypothesis, this maximum must occur in \((a, \sigma^2(b))\). Choose \( c = \max\{c \in (a, \sigma^2(b)) : r(c) = M\} \). Thus

\[
r(t) < r(c), \quad \text{for } c < t < \sigma^2(b).
\]

First show that the point \( c \) cannot be simultaneously left-dense and right-scattered and our working follows that of Akin [4]. Assume the contrary by letting \( \rho(c) = c < \sigma(c) \). If \( r^\Delta(c) \geq 0 \) then \( r(\sigma(c)) \geq r(c) \) and this contradicts (6). If \( r^\Delta(c) < 0 \) then \( \lim_{z \to c^-} r^\Delta(t) = r^\Delta(c) < 0 \). Therefore there exists a \( \delta > 0 \) such that \( r^\Delta(t) < 0 \) on \((c - \delta, c]\). Hence \( r(t) \) is strictly decreasing on \((c - \delta, c]\) and this contradicts the way \( c \) was chosen.

Therefore the point \( c \) cannot be simultaneously left-dense and right-scattered.

By Lemma 1 we must have

\[
r^\Delta(\rho(c)) \leq 0. \tag{7}
\]
Lemma 3 Let $\alpha, \beta \in C_{rd}^2([a, \sigma^2(b)])$ satisfy $\alpha < \beta$,

\begin{align}
\alpha^\Delta &> f(t, y^\sigma), \quad \text{for } y^\sigma \leq \alpha^\sigma \text{ and } t \in [a, b], \\
\beta^\Delta &< f(t, y^\sigma), \quad \text{for } y^\sigma \geq \beta^\sigma \text{ and } t \in [a, b].
\end{align}

(8) (9)

If $y$ is a solution to (1) with $\alpha(a) < y(a) < \beta(a)$ and $\alpha(\sigma^2(b)) < y(\sigma^2(b)) < \beta(\sigma^2(b))$ then $\alpha < y < \beta$ for $t \in [a, \sigma^2(b)]$.

The functions $\alpha$ and $\beta$ satisfying the inequalities in Lemmas 2 and 3 are usually referred to as lower and upper solutions, respectively.

3 Compatibility of Boundary Conditions

In [15] Thompson and in [10] Thompson and Henderson introduced the notion of “compatible boundary conditions” for BVPs when $T = \mathbb{R}$ and $T = \mathbb{Z}$, respectively. The concept of compatible boundary conditions is a simple, degree-based relationship between the given boundary conditions and the lower and upper solutions chosen.

The notion of compatible boundary conditions for BVPs on time-scales will now be naturally extended from the theory of [15] and [10]. The compatibility conditions of [15] and [10] will then follow as special cases of this paper.

In the remainder of this paper assume

$$\Delta = (\alpha(a), \beta(a)) \times (\alpha(\sigma^2(b)), \beta(\sigma^2(b))) \neq \emptyset.$$ 

Definition Call the vector field $\Psi = (\psi^0, \psi^1) \in C(\tilde{\Delta}; \mathbb{R}^2)$ strongly inwardly pointing on $\tilde{\Delta}$ if

\begin{align}
\psi^0(C, D) &> \alpha^\Delta(a), \quad \text{for } C = \alpha(a) \text{ and } \alpha(\sigma^2(b)) \leq D \leq \beta(\sigma^2(b)), \\
\psi^0(C, D) &< \beta^\Delta(a), \quad \text{for } C = \beta(a) \text{ and } \alpha(\sigma^2(b)) \leq D \leq \beta(\sigma^2(b)), \\
\psi^1(C, D) &< \alpha^\Delta(\sigma(b)), \quad \text{for } D = \alpha(\sigma^2(b)) \text{ and } \alpha(a) \leq C \leq \beta(a), \\
\psi^1(C, D) &> \beta^\Delta(\sigma(b)), \quad \text{for } D = \beta(\sigma^2(b)) \text{ and } \alpha(a) \leq C \leq \beta(a).
\end{align}

If we replace the strict inequalities by weak inequalities we say $\Psi$ is inwardly pointing.
Definition Let \( g \in C(\tilde{\Delta} \times \mathbb{R}^2; \mathbb{R}^2) \). Call \( g \) strongly compatible with \( \alpha \) and \( \beta \) if for all strongly inwardly pointing vector fields \( \Psi \) on \( \tilde{\Delta} \),

\[
\mathbb{G}(C, D) \neq (0, 0), \quad \text{for all} \quad (C, D) \in \partial \Delta, \quad (10)
\]

\[
d(G, \Delta, (0, 0)) \neq 0, \quad (11)
\]

where \( \mathbb{G}(C, D) = g((C, D), \Psi(C, D)) \) for all \( (C, D) \in \tilde{\Delta} \). If (10) and (11) hold for all inwardly pointing vector fields \( \Psi \) then we call \( g \) very strongly compatible with \( \alpha \) and \( \beta \).

Remark If \( g \) is (very) strongly compatible with \( \alpha \) and \( \beta \) then the degree (11) is independent of \( \Psi \). It is not difficult to see that strongly inwardly pointing \( \Psi \) always exist. If (10) holds for all inwardly pointing vector fields then we may choose

\[
\psi^0(C, D) = \beta^\Delta(a) \frac{C - \alpha(a)}{\beta(a) - \alpha(a)} + \alpha^\Delta(a) \frac{\beta(a) - C}{\alpha(a) - \beta(a)},
\]

\[
\psi^1(C, D) = \beta^\Delta(\sigma(b)) \frac{D - \alpha(\sigma^2(b))}{\beta(\sigma^2(b)) - \alpha(\sigma^2(b))} + \alpha^\Delta(\sigma(b)) \frac{\beta(\sigma^2(b)) - D}{\alpha(\sigma^2(b)) - \beta(\sigma^2(b))},
\]

when computing (11).

4 Existence of Solutions

In this section some existence results are presented for the BVP (1), (2). The proofs rely on the a priori bounds on solutions of Section 2 and on the following “homotopy principle”, the proof of which can be found in Lloyd [14].

Theorem 4 Let \( H \in K_1(\tilde{D} \times [0, 1]; X) \) such that \( H(z, \lambda) \neq p \) for all \( z \in \partial D \) and all \( \lambda \in [0, 1] \). Then \( d(H(z, \lambda), D, p) \) is independent of \( \lambda \in [0, 1] \).

The homotopy principle above will be applied to the BVP

\[
y^\Delta(t) = l(t, y(\sigma(t))), \quad t \in [a, b], \quad (12)
\]

subject to the boundary conditions (2) via the following lemma.

Lemma 4 Let \( \Omega \times \Delta \subset C([a, \sigma^2(b)]) \times \mathbb{R}^2 \) with \( \Omega \times \Delta \) open and bounded. Let \( H \in K_1(\tilde{\Omega} \times \tilde{\Delta} \times [0, 1]; C([a, \sigma^2(b)]) \times \mathbb{R}^2) \) be such that \( H(y, C, D, 1) = 0 \) is equivalent to the BVP (12), (2). If all solutions \( (y, C, D) \) to \( H(y, C, D, \lambda) = 0 \), satisfy \( (y, C, D) \not\in \partial(\Omega \times \Delta) \) for all \( \lambda \in [0, 1] \) and if \( d(H(y, C, D, 0), \Omega \times \Delta, 0) \neq 0 \) then the BVP (12), (2) has at least one solution.

Proof See that the conditions of Theorem 4 are satisfied and therefore

\[
d(H(\cdot, 1), \Omega \times \Delta, 0) = d(H(\cdot, 0), \Omega \times \Delta, 0) \neq 0.
\]
Hence \( H(y, C, D, 1) = 0 \) has a solution \((y, C, D) \in \Omega \times \triangle\). Since
\[
H(y, C, D, 1) = 0
\]
is equivalent to the BVP \((12), (2)\) then the BVP has at least one solution.

**Theorem 5** Let \( \alpha \leq \beta \in C_r^2([a, \sigma^2(b)]) \) and \( f \in C([a, b] \times \mathbb{R}) \), satisfying
\[
\alpha \Delta \Delta > f(t, y^\sigma), \quad \text{for } y^\sigma < \alpha^\sigma \text{ and } t \in [a, b],
\]
(13)
\[
\beta \Delta \Delta < f(t, y^\sigma), \quad \text{for } y^\sigma > \beta^\sigma \text{ and } t \in [a, b].
\]
(14)
If \( g \in C(\bar{\triangle} \times \mathbb{R}^2; \mathbb{R}^2) \) is strongly compatible with \( \alpha \) and \( \beta \) then the BVP \((1), (2)\) has at least one solution \( y \in C([a, \sigma^2(b)]) \) with \( y \Delta \Delta \in C_r([a, b]) \) satisfying \( \alpha \leq y \leq \beta \) for \( t \in [a, \sigma^2(b)] \).

**Proof**

(i) Modification

Consider the following modified equation with respect to \( \alpha \) and \( \beta \) for each fixed \( t \).
\[
m(t, y^\sigma) = \begin{cases} 
(1 - |J(y^\sigma - \beta^\sigma)|)f(t, \beta^\sigma) + J(y^\sigma - \beta^\sigma)(|f(t, \beta^\sigma)| + 1), & y^\sigma \geq \beta^\sigma, \\
f(t, y^\sigma), & \alpha^\sigma \leq y^\sigma \leq \beta^\sigma, \\
(1 - |J(y^\sigma - \alpha^\sigma)|)f(t, \alpha^\sigma) + J(y^\sigma - \alpha^\sigma)(|f(t, \alpha^\sigma)| + 1), & y^\sigma \leq \alpha^\sigma,
\end{cases}
\]
where \( J \) satisfies
\[
J(t) = \begin{cases} 
1, & \text{for } t \geq 1, \\
t, & \text{for } |t| < 1, \\
-1, & \text{for } t \leq 1.
\end{cases}
\]
Now consider the modified BVP
\[
y \Delta \Delta = m(t, y^\sigma), \quad t \in [a, b],
\]
(15)
\[
(0, 0) = g((y(a), y(\sigma^2(b))); (y \Delta (a), y \Delta (\sigma(b)))).
\]
(16)
The approach now is to show that the BVP \((15), (16)\) has a solution \( y \) satisfying \( \alpha \leq y \leq \beta \) for \( t \in [a, \sigma^2(b)] \). As \( f \) and \( m \) agree in this region then \( y \) will also be the required solution to the BVP \((1), (2)\).

Notice that \( m, \alpha \) and \( \beta \) satisfy
\[
\alpha \Delta \Delta > m(t, y^\sigma), \quad \text{for } y^\sigma < \alpha^\sigma \text{ and } t \in [a, b],
\]
(13)
\[
\beta \Delta \Delta < m(t, y^\sigma), \quad \text{for } y^\sigma > \beta^\sigma \text{ and } t \in [a, b].
\]
(14)
Therefore by Lemma 2, if \( y \) is a solution to \((15)\) and \((y(a), y(\sigma^2(b))) \in \bar{\Delta}\) then \( \alpha \leq y \leq \beta \) for \( t \in [a, \sigma^2(b)] \). Hence \( y \) is the required solution to \((1)\).

(ii) Existence
Consider the function $H(y, C, D, \lambda) = (H_1, H_2) = (0, 0, 0)$ given by

$$
H_1 = \begin{cases} 
    y - 3\lambda w(C, D) - (1 - 3\lambda)(\bar{\alpha} + \bar{\beta})/2, & \text{for } 0 \leq \lambda \leq 1/3, \\
    y - (3\lambda - 1)T(y) - w(C, D), & \text{for } 1/3 \leq \lambda \leq 2/3, \\
    y - T(y) - w(C, D), & \text{for } 2/3 \leq \lambda \leq 1,
\end{cases}
$$

where

$$
\bar{\alpha} = \min_{t \in [a, \sigma^2(b)]} \alpha(t) - 1, \quad \bar{\beta} = \max_{t \in [a, \sigma^2(b)]} \beta(t) + 1,
$$

$$
w(C, D) = \frac{C \sigma^2(b) - Da + (D - C)t}{\sigma^2(b) - a}, \quad \text{for } C, D \in \mathbb{R} \text{ and } a \leq t \leq \sigma^2(b),
$$

$$
T(y) = \int_a^\sigma G(t, s)m(s, y^\sigma)\Delta s, \quad t \in [a, \sigma^2(b)],
$$

where

$$
G(t, s) = \begin{cases} 
    (t - a)(\sigma^2(b) - \sigma(s))/[\sigma^2(b) - a], & \text{for } t \leq s, \\
    (\sigma(s) - a)(\sigma^2(b) - t)/[\sigma^2(b) - a], & \text{for } \sigma(s) \leq t.
\end{cases}
$$

$$
H_2 = \begin{cases} 
    g((C, D); \Psi(C, D)), & 0 \leq \lambda \leq \frac{2}{3}, \\
    g((C, D); (3\lambda - 2)(y^\Delta(a), y^\Delta(\sigma(b))) + 3(1 - \lambda)\Psi(C, D)), & \frac{2}{3} \leq \lambda \leq 1,
\end{cases}
$$

Clearly $H$ is completely continuous and $H(y, C, D, 1) = 0$ is equivalent to the modified BVP (15), (16).

Let $\Omega = \{y \in C([a, \sigma^2(b)]): \bar{\alpha} < y < \bar{\beta} \text{ on } [a, \sigma^2(b)]\}$ and $\Gamma = \Omega \times \Delta$.

Therefore, to apply Lemma 4 we need to show that solutions $(y, C, D)$ to $H$ satisfy $(y, C, D) \notin \partial \Gamma$ for all $\lambda \in [0, 1]$. We investigate the cases $\lambda \in [2/3, 1]$ and $(1/3, 2/3)$; the case $\lambda \in [0, 1/3]$ is trivial.

Case (i) $\lambda \in [2/3, 1]$.

By assumption there is no solution with $\lambda = 1$, so we assume there is a solution $(y, C, D)$ with $\lambda \in [2/3, 1]$. See that $H = 0$ is equivalent to the BVP

$$
y^\Delta = m(t, y^\sigma), \quad t \in [a, b], \quad (17)
y(a) = C, \quad y(\sigma^2(b)) = D, \
(0, 0) = g((C, D); (3\lambda - 2)(y^\Delta(a), y^\Delta(\sigma(b))) + 3(1 - \lambda)\Psi(C, D))).
$$

Now suppose $y$ is a solution of (17) and $(y(0), y(1)) \in \Delta$. By Lemma 2 we have $\alpha \leq y \leq \beta$ for $t \in [a, \sigma^2(b)]$. Hence $y \notin \partial \Omega$.

Assume that $(C, D) \in \partial \Delta$. If $C = y(a) = \alpha(a)$, then $y^\Delta(a) \geq \alpha^\Delta(a)$.

Thus

$$
(3\lambda - 2) y^\Delta(a) + 3(1 - \lambda)\psi^0(y(a), y(\sigma^2(b))) > \alpha^\Delta(a),
$$
since $\Psi$ is strongly inwardly pointing. Similarly, the other cases $(C, D) = (y(a), y(\sigma^2(b))) \in \partial \Delta$ lead to

\begin{align*}
(3\lambda - 2) y^\Delta(a) + 3(1 - \lambda) \psi^0(y(a), y(\sigma^2(b))) &< \beta^\Delta(a), \\
(3\lambda - 2) y^\Delta(\sigma(b)) + 3(1 - \lambda) \psi^1(y(\sigma^1(b)), y(\sigma^2(b))) &< \alpha^\Delta(\sigma(b)), \\
(3\lambda - 2) y^\Delta(\sigma(b)) + 3(1 - \lambda) \psi^1(y(\sigma^2(b)), y(\sigma^2(b))) &> \beta^\Delta(\sigma(b)).
\end{align*}

It follows that

\begin{align*}
(3\lambda - 2)(y^\Delta(a), y^\Delta(\sigma(b))) + 3(1 - \lambda)\Psi(C, D),
\end{align*}

is a strongly inwardly pointing vector field for all $\lambda \in [2/3, 1)$. Since $g$ is strongly compatible,

\[ H_2(y, y(a), y(\sigma^2(b)), \lambda) \neq 0, \]

a contradiction. Thus $(C, D) \not\in \partial \Delta$.

Case (ii) $\lambda \in (1/3, 2/3)$.

See that $H = 0$ is equivalent to the BVP

\begin{align*}
&y^\Delta\Delta = (3\lambda - 1)m(t, y^\sigma), \quad t \in [a, b], \\
&y(a) = C, \quad y(\sigma^2(b)) = D, \\
&(0, 0) = g((C, D); \Psi(C, D)).
\end{align*}

From the boundary conditions we see that $\bar{\alpha}(a) < y(a) < \bar{\beta}(a)$ and $\bar{\alpha}(\sigma^2(b)) < y(\sigma^2(b)) < \bar{\beta}(\sigma^2(b))$. Notice that for $y^\sigma(t) \leq \bar{\alpha}(t)$ and $y^\sigma(t) \geq \bar{\beta}(t)$ we have, respectively

\begin{align*}
(3\lambda - 1)m(t, y^\sigma(t)) = -(3\lambda - 1)(|f(t, y^\sigma(t))| + 1) < 0 = \bar{\alpha}^\Delta(t), \\
(3\lambda - 1)m(t, y^\sigma(t)) = (3\lambda - 1)(|f(t, y^\sigma(t))| + 1) > 0 = \bar{\beta}^\Delta(t).
\end{align*}

Hence Lemma 3 is applicable and $\bar{\alpha} < y < \bar{\beta}$ on $[a, \sigma^2(b)]$. Therefore $y \not\in \partial \Omega$.

Since $\Psi$ is strongly inwardly pointing and $g$ is strongly compatible, by the compatibility conditions there are no solutions $(y, C, D)$ with $(C, D) \in \partial \Delta$.

Thus there are no solutions of $H(y, C, D, \lambda) = 0$ with $(y, C, D) \in \partial \Gamma$ for $\lambda \in [0, 1]$ and $H$ satisfies the conditions of Lemma 4. Therefore, using (3)

\begin{align*}
d(H(\cdot, 1), \Omega \times \Delta, 0) & = d(H(\cdot, 0), \Omega \times \Delta, 0), \\
& = d(y - (\bar{\alpha} + \bar{\beta})/2, \Omega, 0) \times d(G, \Delta, (0, 0)), \\
& = d(G, \Delta, (0, 0)) \neq 0.
\end{align*}

Thus there is a solution $(y, C, D) \in \Gamma$ of $H(y, C, D, 1) = 0$, and hence a solution $y \in C([a, \sigma^2(b)])$ of problem (1) and (2). Since $y$ is continuous and $\sigma$ is right-dense continuous, the composition $y^\sigma$ is right-dense continuous [5]. Since $f$ is continuous we have $y^\Delta(t) = f(t, y^\sigma) \in C_{rd}([a, b])$. This concludes the proof.
Remark There are many variants of Theorem 5 concerning the inequalities involving $\alpha$, $\beta$ and $f$. For example, inequalities (13), (14) may be replaced with

\[
\alpha^{\Delta\Delta} > f(t, \alpha^\sigma), \quad \text{for } t \in [a, b], \\
\beta^{\Delta\Delta} < f(t, \beta^\sigma), \quad \text{for } t \in [a, b],
\]

and the conclusion of Theorem 5 still holds with at least one of the solutions satisfying $\alpha < y < \beta$ for $t \in [a, \sigma^2(b)]$.

As an application of Theorem 5, consider (1) subject to the the Neumann and periodic boundary conditions, respectively

\[
g = (y^{\Delta}(a) - A, y^{\Delta}(\sigma(b)) - B) = (0, 0), \quad \text{(18)}
\]

\[
g = (y(a) - y(\sigma^2(b)), y^{\Delta}(\sigma(b)) - y^{\Delta}(a)) = (0, 0). \quad \text{(19)}
\]

**Lemma 5** Let $\alpha$, $\beta$ and $f$ satisfy the conditions of Theorem 5. If

\[
\beta^{\Delta}(a) \leq A \leq \alpha^{\Delta}(a), \quad \alpha^{\Delta}(\sigma(b)) \leq B \leq \beta^{\Delta}(\sigma(b)),
\]

then the Neumann BVP (1), (18) has at least one solution $y \in C([a, \sigma^2(b)])$ with $y^{\Delta\Delta} \in C_{rad}([a, b])$ satisfying $\alpha \leq y \leq \beta$ for $t \in [a, \sigma^2(b)]$.

**Proof** Assume that (20) holds. Let $\Psi$ be a strongly inwardly pointing vector field on $\Delta = (\alpha(a), \beta(a)) \times (\alpha(\sigma^2(b)), \beta(\sigma^2(b)))$. Here

\[
\mathbb{G}(C, D) = g((C, D), \Psi(C, D)) = (\psi^0(C, D) - A, \psi^1(C, D) - B) = (0, 0).
\]

Since $\Psi$ is strongly inwardly pointing:

\[
\psi^0(\alpha(a), D) - A > \alpha^{\Delta}(a) - A \geq 0, \quad \text{for } \alpha(\sigma^2(b)) \leq D \leq \beta(\sigma^2(b)),
\]

\[
\psi^0(\beta(a), D) - A < \beta^{\Delta}(a) - A \leq 0, \quad \text{for } \alpha(\sigma^2(b)) \leq D \leq \beta(\sigma^2(b)),
\]

\[
\psi^1(C, \alpha(\sigma^2(b))) - B < \alpha^{\Delta}(\sigma(b)) - B \leq 0, \quad \text{for } \alpha(a) \leq C \leq \beta(a),
\]

\[
\psi^1(C, \beta(\sigma^2(b))) - B > \beta^{\Delta}(\sigma(b)) - B \geq 0, \quad \text{for } \alpha(a) \leq C \leq \beta(a).
\]

Hence $\mathbb{G}(C, D) = (\psi^0(C, D) - A, \psi^1(C, D) - B) \neq (0, 0)$ for $(C, D) \in \partial\Delta$.

By continuity of $\Psi$ we must have $d(\mathbb{G}, \triangle(0, 0)) = -1 \neq 0$ and therefore $g$ is strongly compatible. All the conditions of Theorem 5 are satisfied and the result follows.

A simple corollary to Lemma 5 now follows.

**Corollary 1** Let $R > 0$ be a constant. Suppose that $f \in C([a, b] \times \mathbb{R})$ and satisfies

\[
f(t, -R) < 0 < f(t, R), \quad \text{for all } t \in [a, b]. \quad \text{(21)}
\]

Then the homogenous Neumann BVP (1), $(y^{\Delta}(a), y^{\Delta}(\sigma^2(b))) = (0, 0)$ has at least one solution $y \in C([a, \sigma^2(b)])$ with $y^{\Delta\Delta} \in C_{rad}([a, b])$ satisfying $-R < y < R$ on $[a, \sigma^2(b)]$. 
Proof The result follows from Lemma 5 with the choice $-\alpha = \beta = R$.

Lemma 6 Let $\alpha$, $\beta$ and $f$ satisfy the conditions of Theorem 5. If

$$\alpha(a) = \alpha(\sigma^2(b)), \quad \beta(a) = \beta(\sigma^2(b)), \quad \alpha^\Delta(a) \geq \alpha^\Delta(\sigma(b)), \quad \beta^\Delta(a) \leq \beta^\Delta(\sigma(b)), \quad (22)$$

then the periodic BVP $(1)$, $(19)$ has at least one solution $y \in C([a, \sigma^2(b)])$ with $y^\Delta \in C_{rd}([a, b])$ satisfying $\alpha \leq y \leq \beta$ for $t \in [a, \sigma^2(b)]$.

Proof The proof is similar to that of Thompson [15] for the case $T = \mathbb{R}$. Assume that (22) holds. Let $\Psi$ be all strongly inwardly pointing vector fields on $\Delta = (\alpha(a), \beta(a)) \times (\alpha(\sigma^2(b)), \beta(\sigma^2(b)))$. Here

$$\mathbb{G}(C, D) = g((C, D), \Psi(C, D)) = (C - D, \psi^1(C, D) - \psi^0(C, D)) = (0, 0).$$

If $C = \alpha(a) = D$ then $\psi^1(C, D) - \psi^0(C, D) < \alpha^\Delta(\sigma^2(b)) - \alpha^\Delta(a) \leq 0$. If $C = \alpha(a) < D$ then $C - D < 0$ and if $C = \alpha(a) > D$ then $C - D > 0$. Similar inequalities follow for the other cases $(C, D) \in \partial \Delta$. Hence $\mathbb{G} \neq (0, 0)$ on $\partial \Delta$.

Consider the function $H(C, D, \lambda) = (0, 0)$ given by

$$H = \begin{cases} (1 - 2\lambda)\mathbb{G}(C, D) + 2\lambda(C - D, D - \gamma(\sigma^2(b))/2), & 0 \leq \lambda \leq \frac{1}{2}, \\ 2(1 - \lambda)(C - D, D - \gamma(\sigma^2(b))/2) + (2\lambda - 1)(C - \frac{\gamma(a)}{2}, D - \frac{\gamma(\sigma^2(b))}{2}), & \frac{1}{2} \leq \lambda \leq 1. \end{cases}$$

where $\gamma(t) = (\alpha(t) + \beta(t))/2$. See that the conditions of Lemma 4 are satisfied and therefore

$$d(\mathbb{G}, \Delta, (0, 0)) = d(H(\cdot, 0), \Delta, (0, 0)) = d(H(\cdot, 1), \Delta, (0, 0)) = 1 \neq 0.$$

Therefore $g$ is strongly compatible. All the conditions of Theorem 5 are satisfied and the result follows.

Remark In fact, the following two statements can be proven. The Neumann boundary conditions (18) are strongly compatible with $\alpha$ and $\beta$ if and only if (20) holds. The periodic boundary conditions (19) are strongly compatible with $\alpha$ and $\beta$ if and only if (22) holds.

5 BVPs on Infinite Intervals

This section formulates existence theorems for solutions to the following BVP on infinite intervals:

$$y^\Delta(t) = f(t, y^\sigma(t)), \quad t \in [a, \infty), \quad (23)$$

$$g_1(y(a), y^\Delta(a)) = 0, \quad y(t) \text{ is bounded for } t \in [a, \infty). \quad (24)$$

Now we redefine $\Delta = (\alpha(a), \beta(a)) \neq \emptyset$. Let $[a, \infty) = \cup_{k=1}^\infty [a, t_k]$. Throughout this section assume that there exists $t_n \in T$ and $n \in \mathbb{N}$ such that

$$a < t_1 < t_2 < \cdots < t_n < \cdots \quad \text{with} \quad t_n \uparrow \infty \text{ as } n \to \infty.$$
Theorem 6 If the conditions of Theorem 5 hold for each interval \([a, \sigma^2(t_n)]\) then the BVP (23), (24) has at least one solution \(y \in C([a, \infty))\) with \(y^{\Delta\Delta} \in C_{rd}([a, \infty))\) satisfying \(\alpha \leq y \leq \beta\) for \(t \in [a, \infty)\).

Proof Fix \(n \in \mathbb{N}\) and consider the BVP

\[
y^{\Delta\Delta}(t) = f(t, y(\sigma(t))), \quad t \in [a, t_n],
\]

\[
(0, 0) = g((y(a), y^{\Delta}(a)), (y^{\Delta}(a), y^{\Delta}(\sigma(t_n)))) = (g_1(y(a), y^{\Delta}(a)), y - (\alpha(\sigma(t_n)) + \beta(\sigma(t_n)))/2).
\]

It is clear from Theorem 5 that (25), (27) has a solution \(y_n \in C([a, \sigma^2(t_n)])\) with \(\alpha_n \leq y_n \leq \beta_n\) for \(t \in [a, t_n]\). (Note also that \(y_n^{\Delta\Delta} \in C_{rd}([a, \sigma^2(t_n)]\). This argument can be used for each \(n \in \mathbb{N}\). The theorem then follows from Ascoli’s selection theorem [9] applied to a sequence of intervals \([a, t_n]\) as \(n \to \infty\).

6 Acknowledgements

The first Author gratefully acknowledges the financial support of UNSW.

7 References


e-mail: journal@monotone.uwaterloo.ca
http://monotone.uwaterloo.ca/~journal/