Three Point Boundary Value Problems on Time Scales

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(Received 8 August 2003; Revised 5 January 2004; In final form 22 March 2004)

This work formulates existence theorems for solutions to three-point boundary value problems on time scales. The ideas are based on a relationship between the three point boundary conditions, lower and upper solutions and topological degree theory.

Keywords: Time scale; Degree theory; Three-point boundary value problem; Second-order dynamic equation

AMS Subject Classification: 39A12

INTRODUCTION

This paper considers the existence of solutions to the second-order dynamic equation

\[ y^\Delta(t) = f(t, y^\sigma(t)), \quad t \in [a, b], \] (1)

subject to the three point boundary conditions

\[ g((y(a), y(\sigma^2(b)), y(e), (y^\Delta(a), y^\Delta(\sigma(b)))) = (0, 0), \quad a < e < \sigma^2(b), \quad e \in \mathbb{T}, \] (2)

where \( f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}, \) \( g : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) are continuous, and \( t \) is from a so-called “time scale” \( \mathbb{T}. \)

It is assumed that the reader is familiar with the time scale calculus and associated definitions such as delta derivative, jump operators and right-dense continuity. If not, then we refer the reader to Ref. [2].

In Ref. [5], the authors introduced the idea of compatibility of boundary conditions for two point boundary value problems (BVPs) on time scales. In this paper, we extend these ideas to three point BVPs on time scales. The new compatibility conditions are then applied to give some results for the existence of solutions to three point BVPs on time scales. The BVPs treated in this paper include a very wide range of boundary conditions, including nonlinear BVPs. A solution \( y \) to Eq. (1) is a function \( y : [a, \sigma^2(b)] : \rightarrow \mathbb{R} \) satisfying Eq. (1) with \( y \in C^2_{\text{rd}}. \)
COMPATIBILITY OF BOUNDARY CONDITIONS

We will need the two following results, the proofs are found in Ref. [5].

**Lemma 1** Let \( \alpha, \beta \in C^2_{\mathbb{R}}([a, \sigma^2(b)]) \) satisfy \( \alpha(t) \leq \beta(t) \), for \( t \in [a, \sigma^2(b)] \),

\[
\alpha(t) > f(t, u), \quad \text{for } t \in [a, b], \quad u < \alpha''(t), \tag{3}
\]
\[
\beta(t) < f(t, u), \quad \text{for } t \in [a, b], \quad u > \beta''(t). \tag{4}
\]

If \( y \) is a solution to Eq. (1) with \( \alpha(a) \leq y(a) \leq \beta(a) \) and \( \alpha(\sigma^2(b)) \leq y(\sigma^2(b)) \leq \beta(\sigma^2(b)) \),

then \( \alpha(t) \leq y(t) \leq \beta(t) \) for \( t \in [a, \sigma^2(b)] \).

Similarly, the following result holds.

**Lemma 2** Let \( \alpha, \beta \in C^2_{\mathbb{R}}([a, \sigma^2(b)]) \) satisfy \( \alpha(t) < \beta(t) \), \( t \in [a, \sigma^2(b)] \),

\[
\alpha(t) > f(t, u), \quad \text{for } t \in [a, b], \quad u < \alpha''(t), \tag{5}
\]
\[
\beta(t) < f(t, u), \quad \text{for } t \in [a, b], \quad u > \beta''(t). \tag{6}
\]

If \( y \) is a solution to Eq. (1) with \( \alpha(a) < y(a) < \beta(a) \) and \( \alpha(\sigma^2(b)) < y(\sigma^2(b)) < \beta(\sigma^2(b)) \),

then \( \alpha(t) < y(t) < \beta(t) \) for \( t \in [a, \sigma^2(b)] \).

The functions \( \alpha \) and \( \beta \) satisfying the inequalities in Lemmas 1 and 2 are usually referred to as lower and upper solutions, respectively. For more on upper and lower solutions on time scales see Refs. [1, 2 Chapter 6, 3 Chapter 6].

The notion of compatible boundary conditions for BVPs on time-scales will now be naturally extended from the theory in Ref. [5].

In the remainder of this paper assume \( \Delta = (\alpha(a), \beta(a)) \times (\alpha(\sigma^2(b)), \beta(\sigma^2(b))) \neq \emptyset \).

**Definition 1** We say that the vector field \( \Psi = (\psi^0, \psi^1) \in C(\overline{\Delta}; \mathbb{R}^2) \) is strongly inwardly pointing on \( \Delta \) if

\[
\psi^0(C, D) > \alpha^2(a) \quad \text{for } C = \alpha(a), \quad \text{and } \alpha(\sigma^2(b)) \leq D \leq \beta(\sigma^2(b)),
\]
\[
\psi^1(C, D) < \beta^2(a) \quad \text{for } C = \beta(a), \quad \text{and } \alpha(\sigma^2(b)) \leq D \leq \beta(\sigma^2(b)),
\]
\[
\psi^0(D, C) < \alpha^2(\sigma^2(b)) \quad \text{for } D = \alpha(\sigma^2(b)), \quad \text{and } \alpha(a) \leq C \leq \beta(a),
\]
\[
\psi^1(D, C) > \beta^2(\sigma^2(b)) \quad \text{for } D = \beta(\sigma^2(b)), \quad \text{and } \alpha(a) \leq C \leq \beta(a).
\]

If we replace the strict inequalities by weak inequalities, then we say \( \Psi \) is inwardly pointing.

**Definition 2** Let \( g \in C(\overline{\Delta}; \mathbb{R} \times \mathbb{R}^2; \mathbb{R}^2) \). We say that \( g \) is strongly compatible with \( \alpha \) and \( \beta \) if for all strongly inwardly pointing vector fields \( \Psi \) on \( \Delta \) and all continuous functions \( \phi : [\alpha(a), \beta(a)] \to [\alpha(e), \beta(e)] \),

\[
\varnothing(C, D) \neq (0, 0), \quad \text{for all } (C, D) \in \partial \Delta, \tag{7}
\]
\[
dG(C, \Delta, (0, 0)) \neq 0, \tag{8}
\]

where \( \varnothing(C, D) = g((C, D), \phi(C), \Psi(C, D)) \) for all \( (C, D) \in \overline{\Delta} \) and \( dG(C, \Delta, (0, 0)) \) is the degree of \( \varnothing \) at \( (0, 0) \) relative to \( \Delta \). If Eqs. (7) and (8) hold for all inwardly pointing vector fields \( \Psi \) and all \( \phi \) then we call \( g \) very strongly compatible with \( \alpha \) and \( \beta \).


EXISTENCE OF SOLUTIONS

In this section, some existence results are presented for the BVP (1) and (2). The proofs rely on the \textit{a priori} bounds on solutions of \textquotedblleft Compatibility of Boundary Conditions Section\textquotedblright and on the following \textit{\textquotedblleft Homotopy Principle\textquotedblright}, the proof of which can be found in Ref. [4]. Assume $E$ is a bounded, open subset of the normed space $X$ and $p \in X$. See Ref. [4] for the definition of $K_1$ in Theorem 1.

**Theorem 1** Let $H \in K_1(\partial E \times [0, 1]; X)$ such that $H(z, \lambda) \neq p$ for all $z \in \partial E$ and all $\lambda \in [0, 1]$. Then $d(H(z, \lambda), D, p)$ is independent of $\lambda \in [0, 1]$.

The Homotopy principle above will be applied to the modified BVP

$$y^\Delta = m(t, y^\sigma), \quad t \in [a, b],$$  

(9)

where

$$m(t, u) = \begin{cases} 
(1 - |J(u - \beta^\sigma(t))|)f(t, \beta^\sigma(t)) + J(u - \beta^\sigma(t))(|f(t, \beta^\sigma(t)) + 1|, \quad u \geq \beta^\sigma(t), \\
(1 - |J(u - \alpha^\sigma(t))|)f(t, \alpha^\sigma(t)) + J(u - \alpha^\sigma(t))(|f(t, \alpha^\sigma(t)) + 1|, \quad u \leq \alpha^\sigma(t), \\
& \text{and } J \text{ is given by} \\
J(v) = \begin{cases} 
1, & v \geq 1, \\
v, & |v| < 1, \\
-1, & v \leq 1,
\end{cases}
\end{cases}$$

subject to the boundary conditions (2) via the following lemma.

**Lemma 3** Let $\Omega \times \Delta \subset C([a, \sigma^2(b)]) \times \mathbb{R}^2$ with $\Omega \times \Delta$ open and bounded. Let $H \in K_1(\Omega \times \Delta \times [0, 1]; C([a, \sigma^2(b)]) \times \mathbb{R}^2)$ be such that $H(y, C, D, \lambda) = 0$ is equivalent to the BVP (9) and (2). If all solutions $(y, C, D)$ to $H(y, C, D, \lambda) = 0$, satisfy $(y, C, D) \not\in \partial(\Omega \times \Delta)$ for all $\lambda \in [0, 1]$ and if $d(H(y, C, D, 0), \Omega \times \Delta, 0) \neq 0$, then the BVP (9) and (2) has at least one solution.

**Proof** Note that the conditions of Theorem 1 are satisfied and therefore

$$d(H(\cdot, 1), \Omega \times \Delta, 0) = d(H(\cdot, 0), \Omega \times \Delta, 0) \neq 0.$$ 

Hence $H(y, C, D, 1) = 0$ has a solution $(y, C, D) \in \Omega \times \Delta$. Since $H(y, C, D, 1) = 0$ is equivalent to the BVP (9) and (2) the BVP has at least one solution. \qed

**Theorem 2** Assume $\alpha, \beta \in C^2_{\text{ad}}([a, \sigma^2(b)])$ with $\alpha(t) \leq \beta(t)$ for $t \in [a, \sigma^2(b)]$ and $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous. Further assume

$$\alpha^\Delta(t) > f(t, u), \quad \text{for } t \in [a, b], \quad u < \alpha^\sigma(t), \quad (10)$$

$$\beta^\Delta(t) < f(t, u), \quad \text{for } t \in [a, b], \quad u > \beta^\sigma(t). \quad (11)$$

If $g \in C(\Delta \times \mathbb{R} \times \mathbb{R}^2; \mathbb{R}^2)$ is strongly compatible with $\alpha$ and $\beta$ then the BVP (1) and (2) has at least one solution $y \in C([a, \sigma^2(b)])$ with $y^\Delta \in C^2_{\text{ad}}([a, b])$ satisfying $\alpha(t) \leq y(t) \leq \beta(t)$ for $t \in [a, \sigma^2(b)]$.
Proof \( (i) \) Modification

Consider the following modified equation of (9) with respect to \( \alpha \) and \( \beta \)
\[
y = m(t,y^\alpha), \quad t \in [a,b],
\]
(12)
\[
(0,0) = g((y(a),y(\sigma^2(b)),y(e)),(y^\Delta(a),y^\Delta(\sigma(b)))).
\]
(13)

The approach now is to show that the BVP (12) and (13) has a solution \( y \) satisfying \( \alpha(t) \leq y(t) \leq \beta(t) \) for \( t \in [a, \sigma^2(b)] \). As \( f \) and \( m \) agree in this region then \( y \) will also be the required solution to the BVP (1) and (2).

Notice that \( m, \alpha \) and \( \beta \) satisfy
\[
\alpha(t) > m(t,u), \quad \text{for } t \in [a,b], \quad u < \alpha^*(t),
\]
\[
\beta(t) < m(t,u), \quad \text{for } t \in [a,b], \quad u > \beta^*(t).
\]

Therefore by Lemma 1, if \( y \) is a solution to Eq. (12) and \( (\gamma(a), \gamma(\sigma^2(b))) \in \tilde{\Delta} \) then \( \alpha(t) \leq y(t) \leq \beta(t) \) for \( t \in [a, \sigma^2(b)] \). Hence \( y \) is the required solution to Eq. (1).

(ii) Existence

Consider the equation \( H(y,C,D,\lambda) = (H_1(y,C,D,\lambda), H_2(y,C,D,\lambda)) = (0,0,0) \), where
\[
H_1(y,C,D,\lambda) = \begin{cases} 
  y - 3\lambda w(C,D) - (1 - 3\lambda)(\alpha + \beta)/2, & \text{for } 0 \leq \lambda \leq 1/3, \\
  y - (3\lambda - 1)/2(y) - w(C,D), & \text{for } 1/3 \leq \lambda \leq 2/3, \\
  y - Ty - w(C,D), & \text{for } 2/3 \leq \lambda \leq 1,
\end{cases}
\]
where
\[
\alpha = \min_{r \in [a,\sigma^2(b)]} \alpha(r) - 1, \quad \beta = \max_{r \in [a,\sigma^2(b)]} \beta(r) + 1,
\]
\[
w(C,D)(t) = \frac{Ca^2(b) - Da + (D - C)t}{\sigma^2(b) - a}, \quad \text{for } C, D \in \mathbb{R} \quad \text{and } a \leq t \leq \sigma^2(b),
\]
\[
(Ty)(t) = \int_{\sigma(s)}^{\sigma(b)} G(t,s)m(s,y^\alpha(s)) \Delta s, \quad t \in [a, \sigma^2(b)],
\]
where
\[
G(t,s) = \begin{cases} 
  \frac{-(t-s)(\sigma^2(b)-\sigma(s))}{\sigma^2(b)-a}, & \text{for } t \leq s, \\
  \frac{(\sigma(s)-\sigma)(\sigma^2(b)-a)}{\sigma^2(b)-a}, & \text{for } \sigma(s) \leq t.
\end{cases}
\]

and
\[
H_2(y,C,D,\lambda) =
\begin{cases} 
  g((C,D), \phi(C), \Psi(C,D)), & \text{for } 0 \leq \lambda \leq 2/3, \\
  g(C,D), (3\lambda - 2)y(e) + 3(1 - \lambda)\phi(C), (3\lambda - 2)(y^\Delta(a),y^\Delta(\sigma(b))) + 3(1 - \lambda)\Psi(C,D)), & \text{for } 2/3 \leq \lambda \leq 1,
\end{cases}
\]
Clearly $H$ is completely continuous and $H(y, C, D, 1) = 0$ is equivalent to the modified BVP (12) and (13).

Let $\Omega = \{ y \in C([a, \sigma^2(b)]) : \bar{a} < y(t) < \bar{\beta} \text{ on } [a, \sigma^2(b)] \}$ and $\Gamma = \Omega \times \Delta$.

Therefore, to apply Lemma 3 we need to show that solutions $(y, C, D)$ to $H(y, C, D, \lambda) = 0$ satisfy $(y, C, D) \not\in \partial \Gamma$ for all $\lambda \in [0, 1]$. We investigate the cases $\lambda \in [2/3, 1]$ and $(1/3, 2/3)$; the case $\lambda \in [0, 1/3]$ is trivial.

Case (i) $\lambda \in [2/3, 1]$.

By assumption there is no solution with $\lambda = 1$, so we assume there is a solution $(y, C, D)$ with $\lambda \in [2/3, 1)$. Note that $H(y, C, D, \lambda) = 0$ is equivalent to the BVP

$$ y^{\Delta \lambda} = m(t, y''), \quad t \in [a, b], $$

$$ y(a) = C, \quad y(\sigma^2(b)) = D, $$

$$(0, 0) = g((C, D), (3\lambda - 2)y(e) + 3(1 - \lambda)\phi(C), (3\lambda - 2)(y^{\Delta}(a), y^{\Delta}(\sigma(b)) + 3(1 - \lambda)\Psi(C, D))).$$

Now suppose $y$ is a solution of Eq. (14) and $(y(a), y(\sigma^2(b))) \in \bar{\Delta}$. By Lemma 1 we have $\alpha(t) \leq y(t) \leq \beta(t)$ for $t \in [a, \sigma^2(b)]$. Hence $y \not\in \partial \Omega$.

Assume that $(C, D) \in \partial \Delta$. If $C = y(a) = \alpha(a)$, then $y^{\Delta}(a) \geq \alpha^{\Delta}(a)$. Thus

$$(3\lambda - 2)y^{\Delta}(a) + 3(1 - \lambda)\psi^0(y(a), y(\sigma^2(b))) > \alpha^{\Delta}(a),$$

since $\Psi$ is strongly inwardly pointing. Similarly, the other cases $(C, D) = (y(a), y(\sigma^2(b))) \in \partial \Delta$ lead to

$$(3\lambda - 2)y^{\Delta}(a) + 3(1 - \lambda)\psi^0(y(a), y(\sigma^2(b))) < \beta^{\Delta}(a),$$

$$(3\lambda - 2)y^{\Delta}(a) + 3(1 - \lambda)\psi^1(y(\sigma(b)), y(\sigma^2(b))) < \beta^{\Delta}(\sigma(b)),$$

$$(3\lambda - 2)y^{\Delta}(a) + 3(1 - \lambda)\psi^1(y(\sigma(b)), y(\sigma^2(b))) > \beta^{\Delta}(\sigma(b)).$$

It follows that

$$(3\lambda - 2)(y^{\Delta}(a), y^{\Delta}(\sigma(b))) + (3 - \lambda)\Psi(C, D),$$

is a strongly inwardly pointing vector field for all $\lambda \in [2/3, 1)$. Since $g$ is strongly compatible,

$$H_2(y, y(a), y(\sigma^2(b)), \lambda) \neq 0,$$

a contradiction. Thus $(C, D) \not\in \partial \Delta$.

Case (ii) $\lambda \in (1/3, 2/3)$.

Note that $H(y, C, D, \lambda) = 0$ is equivalent to the BVP

$$ y^{\Delta \lambda} = (3\lambda - 1)m(t, y''), \quad t \in [a, b], $$

$$ y(a) = C, \quad y(\sigma^2(b)) = D, \quad g((C, D), \phi(C), \Psi(C, D)) = (0, 0).$$

Let $\bar{\alpha}(t) = \bar{a}$ and $\bar{\beta}(t) = \bar{\beta}$, for $t \in [a, \sigma^2(b)]$. From the boundary conditions we see that $\bar{\alpha}(a) < y(a) < \bar{\beta}(a)$ and $\bar{\alpha}(\sigma^2(b)) < y(\sigma^2(b)) < \bar{\beta}(\sigma^2(b))$. Notice that for $u \leq \bar{\alpha}'(t) = \bar{a}$
and $u \geq \beta^\sigma(t) = \tilde{\beta}$ we have, respectively

$$(3\lambda - 1)m(t, u) = -(3\lambda - 1)(|f(t, u)| + 1) < 0 = \tilde{\alpha}^\Delta(t),$$

$$(3\lambda - 1)m(t, u) = (3\lambda - 1)(|f(t, u)| + 1) > 0 = \tilde{\beta}^\Delta(t).$$

Hence Lemma 2 is applicable and $\tilde{\alpha}(t) = \alpha < y(t) < \tilde{\beta}(t) = \beta$ on $[a, \sigma^2(b)]$. Therefore $y \notin \partial\Omega$.

Since $\Psi$ is strongly inwardly pointing and $g$ is strongly compatible, by the compatibility conditions there are no solutions $(y, C, D)$ with $(C, D) \in \partial\Delta$.

Thus there are no solutions of $H(y, C, D, \lambda) = 0$ with $(y, C, D) \in \partial\Gamma$ for $\lambda \in [0, 1]$ and $H$ satisfies the conditions of Lemma 3. Therefore,

$$d(H(\cdot, 1), \Omega \times \Delta, 0) = d(H(\cdot, 0), \Omega \times \Delta, 0),$$

$$= d(y - (\tilde{\alpha} + \tilde{\beta})/2, \Omega, 0) \times d(\tilde{\Omega}, \Delta, (0, 0),$$

$$= d(\tilde{\Omega}, \Delta, (0, 0)) \neq 0.$$

Thus there is a solution $(y, C, D) \in \Gamma$ of $H(y, C, D, 1) = 0$, and hence a solution $y \in C([a, \sigma^2(b)])$ of problem (1) and (2). Since $y$ is continuous and $\sigma$ is right-dense continuous, the composition $y^\sigma$ is right-dense continuous [2]. Since $f$ is continuous we have $y^\Delta = f(t, y^\sigma) \in C_{rd}([a, b])$. This concludes the proof. 

**Remark** There are many variants of Theorem 2 concerning the inequalities involving $\alpha$, $\beta$ and $f$. For example, inequalities (10) and (11) may be replaced with

$$\alpha^\Delta(t) > f(t, \alpha^\sigma(t)), \quad \text{for } t \in [a, b],$$

$$\beta^\Delta(t) < f(t, \beta^\sigma(t)), \quad \text{for } t \in [a, b],$$

and the conclusion of Theorem 2 still holds with at least one of the solutions satisfying $\alpha(t) < y(t) < \beta(t)$ for $t \in [a, \sigma^2(b)]$.

**Remark** Existence theorems for $m > 3$ point BVPs follow by extending the notion of compatibility through the introduction of appropriate functions $\phi_i$ for $i = 1, \ldots, m - 2$ such that each $\phi_i : [a(a), \beta(a)] \to [a(e_i), \beta(e_i)]$ and the compatibility conditions hold.

**Acknowledgements**

The first author greatly appreciates the support of NSF Grant 0072505 and the third author gratefully acknowledges the financial support of UNSW.

**References**