Three-Point Boundary Value Problems for Second-Order Discrete Equations

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Abstract—We formulate existence results for solutions to discrete equations which approximate three-point boundary value problems for second-order ordinary differential equations. © 2003 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

We consider the three-point boundary value problem

\[ y'' = f(x, y, y'), \quad 0 \leq x \leq 1, \]
\[ (0, 0) = G(y(0), y(1), y(a)), \quad 0 < a < 1, \]

and its discrete approximation

\[ D^2 y_{k+1} = f(x_k, y_k, D y_k), \quad k = 1, \ldots, n - 1, \]
\[ (0, 0) = G(y_0, y_n, y_d), \quad d \in \{1, \ldots, n - 1\}, \]

where \( f \) and \( G \) are continuous and perhaps nonlinear. The step size \( h = 1/n \), the grid points \( x_k = kh \) for \( k = 0, \ldots, n \), and \( D y_k = (y_k - y_{k-1})/h \) for \( k = 1, \ldots, n \), so that \( D^2 y_{k+1} = (y_{k+1} - 2y_k + y_{k-1})/h^2 \) for \( k = 1, \ldots, n - 1 \).

By a solution to (1) we mean a real-valued, twice continuously differentiable function \( y(x) \) satisfying (1) for all \( x \in [0, 1] \).

By a solution to (3) we mean a vector \( \bar{y} = (y_0, \ldots, y_n) \in \mathbb{R}^{n+1} \) satisfying (3) for all \( k = 1, \ldots, n - 1 \). The value of the \( k \)th component, \( y_k \), of a solution \( \bar{y} \) of (3) is expected to approximate \( y(x_k) \), for some solution \( y \) of (1).
Much research has been conducted on the continuous problem (1), (2) with three- (or more) point boundary value problems enjoying interest in [1–14].

Less is known about the corresponding discrete three-point problem (3), (4); however, two-point discrete boundary value problems have been studied in [15–18]. To our knowledge, no research exists regarding the existence of solutions to discrete, second-order, multipoint boundary value problems. This paper’s purpose is to fill this gap in the literature.

Three-point boundary conditions can arise in, for example, the bending of a beam where conditions may be imposed at the ends of the beam, as well as at an interior point to improve stability or for other reasons (see [1]).

Besides discretely approximating the occurrences mentioned above, the motivation for our investigation arises from the fundamental changes which occur when a boundary value problem is discretised. Although the continuous problem may admit a solution, the existence of a solution to the discrete problem is far from guaranteed (see [19, 20]).

Thompson’s notion of compatibility for two-point boundary conditions [21], which is a relationship between the given boundary conditions and the lower and upper solutions chosen, is extended to discrete three-point boundary conditions. We show that, for a small step size, if the continuous three-point boundary conditions are strongly compatible, then the discrete three-point boundary conditions are discrete strongly compatible. Compatibility for (2) has been extensively discussed in [13, 22].

Once these new discrete compatibility conditions are defined, the existence of solutions to (3), (4) follows immediately.

We remark that our results extend to different types of three- (or more) point boundary conditions (for example, those boundary conditions involving derivatives at the end-points) once the suitable compatibility conditions are introduced.

2. NOTATION AND PRELIMINARY RESULTS

For a set $U$, let $\partial U$ denote the boundary of $U$ and let $\bar{U}$ denote its closure.

Denote the space of $m$ times continuously differentiable functions mapping from $A$ to $B$ by $C^m(A; B)$ endowed with the usual maximum norm. If $B = \mathbb{R}$, then we omit the $B$. For $\tilde{y}, \tilde{z} \in \mathbb{R}^{n+1}$ write $\tilde{y} \leq \tilde{z}$ if $y_i \leq z_i$ for all $i = 0, \ldots, n$ and denote $\|\tilde{y}\| = \max_{k=0, \ldots, n} |y_k|$. We say $\tilde{y}$ is a constant vector if each component of $\tilde{y}$ is identically equal to some $b \in \mathbb{R}$. If $g \in C([0, 1])$, then $\hat{g} = (g(0), \ldots, g(nh))$.

If $U$ is a bounded, open subset of $\mathbb{R}^n$, $q \in \mathbb{R}^n$, $F \in C(\bar{U}; \mathbb{R}^n)$, and $q \notin F(\partial U)$, we denote the corresponding Brouwer degree of $F$ on $U$ at $q$ by $d(F, U, q)$.

Modification of $f$ is common practice for existence proofs of boundary value problems and we will make the necessary modifications by using the following functions.

**Definition 1.** If $a \leq b$ are given, let $\pi : \mathbb{R} \to [a, b]$ be (the retraction) given by $\pi(y, a, b) = \max\{\min\{b, y\}, a\}$. Let $K \in C(\mathbb{R}; [-1, 1])$ satisfy

(i) $K(t) = 1$, \quad $t > 1$,

(ii) $K(t) = t$, \quad $-1 \leq t \leq 1$,

(iii) $K(t) = -1$, \quad $t < -1$.

If $\alpha \leq \beta$ are given, let $T \in C(\mathbb{R})$ be given by $T(y, \alpha, \beta) = K(y - \pi(y, \alpha, \beta))$. Define a function $l \in C([0, 1] \times \mathbb{R}^2)$ by

\[ l(x, y, y') = f(x, y, \pi(y', -L, L)), \quad \text{and let} \]

\[ m(x, y, y') = \left(1 - |T(y(x), \alpha(x), \beta(x))|\right) l(x, \pi(y, \alpha, \beta), y') + T(y(x), \alpha(x), \beta(x)) \left(|l(x, \pi(y, \alpha, \beta), y')| + 1\right), \]

where $L > 0$ is chosen from Lemma 4 below.
DEFINITION 2. Call $\alpha$ (or $\beta$) a strict lower (strict upper) solution for (1) if $\alpha$ (or $\beta$) $\in C^2([0, 1])$

$$
\alpha''(x) - f(x, \alpha(x), \alpha'(x)) \geq \gamma, \quad \beta''(x) - f(x, \beta(x), \beta'(x)) \leq \gamma,
$$

for some $\gamma > 0$ and all $x \in [0, 1]$. We shall refer to the pair $\alpha$, $\beta$ as nondegenerate if $\Delta = \{(\alpha(0), \beta(0)) \times (\alpha(1), \beta(1)) \neq \emptyset$, i.e., $\alpha(0) < \beta(0)$ and $\alpha(1) > \beta(1)$.

We assume there exist nondegenerate strict lower and strict upper solutions $\alpha$ and $\beta$ with $\alpha \leq \beta$. In what follows, $\bar{\alpha} = (\alpha(0), \alpha(h), \ldots, \alpha(nh))$ where $\alpha$ is a strict lower solution for (1). We define $\bar{\beta}$ similarly. Let $\alpha_m = \min \alpha(x)$ and $\beta_M = \max \beta(x)$.

3. SOME DISCRETE NAGUMO CONDITIONS

We now state the main conditions for guaranteeing a priori bounds on first difference quotients of solutions to (3). These estimates are independent of the step size and are needed in the proofs of our convergence theorems later in the work.

LEMMA 1. DISCRETE NAGUMO. If there exists a continuous, positive, real-valued, nondecreasing function $\Psi$ such that $\bar{y} \in \mathbb{R}^{n+1}$ satisfies $\bar{\alpha} \leq \bar{y} \leq \bar{\beta}$ and

$$
|\mathcal{D}y_{k+1}| \leq \Psi(|\mathcal{D}y_k|) + 1, \quad \text{for } k = 1, \ldots, n,
$$

with

$$
\int_0^\infty s \frac{ds}{\Psi(s)} = \infty,
$$

then there exists a positive constant $N$ (depending on $\alpha$, $\beta$, and $\Psi$) such that $|\mathcal{D}y_k| \leq N$ for $k = 1, \ldots, n$.

PROOF. See [17].

4. COMPATIBILITY OF BOUNDARY CONDITIONS

In this section, we extend the notion of discrete compatibility, which is a degree-based relationship between the given boundary conditions and the lower and upper solutions chosen, to three-point boundary conditions.

DEFINITION 3. Let $G \in C(\bar{\Delta} \times \mathbb{R}; \mathbb{R}^2)$. We say $G$ is strongly compatible with $\alpha$ and $\beta$ if for all continuous functions $\phi_\alpha : [\alpha(0), \beta(0)] \to [\alpha(a), \beta(a)]$,

$$
\mathcal{G}(C, D) \neq (0, 0), \quad \text{for all } (C, D) \in \partial \Delta, \quad \text{and}
$$

$$
d(\mathcal{G}, \alpha, (0, 0)) \neq 0,
$$

where $\mathcal{G}(C, D) = G(C, D, \phi_\alpha(C))$ for all $(C, D) \in \bar{\Delta}$ and $\alpha$ is given in (2).

DEFINITION 4. Let $G \in C(\bar{\Delta} \times \mathbb{R}; \mathbb{R}^2)$. We say $G$ is discrete strongly compatible with $\bar{\alpha}$ and $\bar{\beta}$ if for all continuous functions $\phi_d : [\alpha_0, \beta_0] \to [\alpha_d, \beta_d]$,

$$
\mathcal{G}(C, D) \neq (0, 0), \quad \text{for all } (C, D) \in \partial \Delta, \quad \text{and}
$$

$$
d(\mathcal{G}, \alpha, (0, 0)) \neq 0,
$$

where $\mathcal{G}(C, D) = G(C, \phi_d(C), D)$ for all $(C, D) \in \bar{\Delta}$ and $d$ is given in (4).

LEMMA 2. Consider $G(y(0), y(1), y(a)) = (0, 0)$, with $0 < a < 1$. Let $G$ be continuous and strongly compatible with $\alpha$ and $\beta$. Then there exists a $\delta > 0$ such that for $b$ satisfying $0 < b < 1$ and $|a - b| < \delta$, then the boundary conditions $G(y(0), y(1), y(b)) = (0, 0)$ are strongly compatible with $\alpha$ and $\beta$.

PROOF. For $0 \leq c \leq 1$, let $\Phi_c = C(\bar{\Delta}, [\alpha(c), \beta(c)])$ and $\mathcal{G}_c(C, D) = G(C, D, \phi_c(C))$ for $(C, D) \in \bar{\Delta}$. Let

$$
e = \inf_{\phi_\alpha \in \Phi_c, (C, D) \in \partial \Delta} \|\mathcal{G}_c(C, D)\|.$$

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By uniform continuity of $G$ on $\bar{\Delta} \times [\alpha_m, \beta_M]$ and of $\alpha$, $\beta$ on $[0,1]$, choose $\delta > 0$ such that if $0 < b < 1$, $|a - b| < \delta$, and $\alpha(b) \leq y \leq \beta(b)$, then

$$\max_{(C, D) \in \partial \Delta} \|G(C, D, y) - G(C, D, z)\| < \varepsilon,$$

for any $z$ such that $|y - z| < \delta$, $\alpha(a) \leq z \leq \beta(a)$, $|\alpha(a) - \alpha(b)| < \delta/2$, and $|\beta(a) - \beta(b)| < \delta/2$. Let $\kappa : [\alpha(b) - \delta/2, \beta(b) + \delta/2] \rightarrow [\alpha(a), \beta(a)]$ be increasing, linear, and onto. For each $\phi_b \in \Phi_b$, it suffices to show that $d(\mathcal{G}_b, \Delta, (0,0)) \neq 0$. Now $\phi_a = \kappa \circ \phi_b \in \Phi_a$ and

$$\max_{\alpha(0) \leq C \leq \beta(0)} |\phi_a(C) - \phi_b(C)| < \delta,$$

so

$$\max_{(C, D) \in \partial \Delta} |\mathcal{G}_a(C, D) - \mathcal{G}_b(C, D)| < \varepsilon.$$

Thus, $d(\mathcal{G}_b, \Delta, (0,0)) = d(\phi_a, \Delta, (0,0)) \neq 0$ as required.

**Lemma 3.** Let (2) be strongly compatible with $\alpha$, $\beta$ and let $\delta$ be given in Lemma 2. If $0 < h < \delta$, then (4) is discrete strongly compatible with $\bar{\alpha}$, $\bar{\beta}$ when $d \in \{1, \ldots, n-1\}$ satisfies $|a - dh| < h$.

**Proof.** In the previous proof, replace $b$ with $dh$ where $d \in \{1, \ldots, n-1\}$ is such that $|a - dh| < h < \delta$. We see from the proof of Lemma 2 that $d(\mathcal{G}_d, \Delta, (0,0)) = d(\mathcal{G}_a, \Delta, (0,0)) \neq 0$. Thus, equation (2) is discrete strongly compatible with $\bar{\alpha}$ and $\bar{\beta}$ as required.

**5. Existence Results**

**Lemma 4.** Let $\alpha \leq \beta$ be strict lower and strict upper solutions for (1) and let $m(x, y, p)$ be given in Definition 1. There exists a $\delta_2 \in (0, \delta)$ such that if $0 < h < \delta_2$, then every solution $\bar{y}$ of

$$m(x_k, y_k, D_y k), \quad k = 1, \ldots, n-1,$$

with $(y_0, y_n) \in \Delta$ satisfies $\bar{\alpha} \leq \bar{y} \leq \bar{\beta}$.

**Proof.** (See [17].) We may choose $\delta_2 > 0$ sufficiently small such that $\bar{\alpha}$ and $\bar{\beta}$ are “strict discrete lower” and “strict discrete upper” solutions, respectively, for (3), where $N$ is given in Lemma 1, $L \geq N$ is chosen such that $\max\{|D_\alpha_k|, |D_\beta_k| : k = 1, \ldots, n\} \leq L$. Note that $\|m(x, y, p)\| \leq \Psi(\|p\|) + 1$ for all $(x, y, p) \in [0,1] \times \mathbb{R} \times \mathbb{R}$.

We now present our main result.

**Theorem 1.** Assume there exist nondegenerate strict lower and strict upper solutions $\alpha \leq \beta$ for (1) and that $f$ satisfies

$$\|f(x, y, p)\| \leq \Psi(\|p\|), \quad x \in [0,1], \quad \alpha \leq y \leq \beta, \quad p \in \mathbb{R},$$

where $\Psi$ satisfies the conditions of Lemma 1. If $G \in C(\bar{\Delta} \times \mathbb{R}; \mathbb{R}^2)$ is strongly compatible with $\alpha$ and $\beta$, then there exists a $\delta_2 > 0$ such that for $0 < h < 1/n < \delta_2$ there exists a solution $\bar{y}$ of problem (3),(4) with $\bar{\alpha} \leq \bar{y} \leq \bar{\beta}$.

**Proof.** Consider

$$D^2 y_{k+1} = m(x_k, y_k, D_y k), \quad k = 1, \ldots, n-1,$$

and (4) where $\delta_2$ is given in Lemma 4. From Lemma 4, see that there is a $\delta_2 > 0$ such that for $0 < h < \delta_2$, any solution $\bar{y}$ to (7) with $(y_0, y_n) \in \Delta$ satisfies $\bar{\alpha} \leq \bar{y} \leq \bar{\beta}$; since $m$ satisfies

$$\|m(x, y, p)\| \leq \Psi(\|p\|) + 1, \quad \text{for all } (x, y, p) \in [0,1] \times \mathbb{R} \times \mathbb{R},$$

and by Lemma 1 there is $N > 0$ such that $|D_y k| \leq N$ for $k = 1, \ldots, n$. Thus, any solution to (7) is also a solution to (3).
Define an operator \( T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) by
\[
T(y)_k = h \sum_{i=1}^{n+1} Q(x_i, s_i)m(s_i, y_i, Dy_i) + C(1 - x_k) + Dx_k, \quad k = 0, \ldots, n,
\]
where
\[
Q(t, s) = \begin{cases} 
(s - 1)t, & \text{for } 0 \leq t \leq s \leq 1, \\
(t - 1)s, & \text{for } 0 \leq s \leq t \leq 1.
\end{cases}
\]
Choose \( a < 0 \) and \( b > 0 \) such that \( a \leq \min \bar{a} - 1 \) and \( b \geq \max \bar{b} + 1 \). Let
\[
\Omega = \{ \bar{y} \in \mathbb{R}^{n+1} : \bar{a} < \bar{y} < \bar{b}, \|Dy_k\| < N + 1, k = 1, \ldots, n \},
\]
and let \( \triangle \) be given as in Definition 2. Let \( \phi_d \) be as in Definition 4. Now, for \( \lambda \in [0, 1] \), consider
\[
((I - \lambda T)(\bar{y}), G(C, D, \lambda y_d + (1 - \lambda)\phi_d(C)) = (0, (0, 0)). \tag{9}
\]
This is equivalent to \( (\bar{y}, C, D) \) satisfying
\[
D^2y_{k+1} = \lambda m(x_k, y_k, Dy_k), \quad k = 1, \ldots, n - 1, \tag{10}
\]
\[
y_0 = \lambda C, \quad y_n = \lambda D, \quad \text{and} \tag{11}
\]
\[
(0, 0) = G(C, D, \lambda y_d + (1 - \lambda)\phi_d(C)). \tag{12}
\]
The problem is reduced to showing there is a \( (\bar{y}, C, D) \in \Omega \times \triangle \) such that (9) holds.

Assume \( (\bar{y}, C, D) \in \overline{\Omega} \times \triangle \). We show that if (9) holds, then \( (\bar{y}, C, D) \in \Omega \times \triangle \) (and consequently, \( (\bar{y}, C, D) \notin \partial(\Omega \times \triangle) \)). First, see that this is trivially satisfied for \( \lambda = 0 \) so assume \( \lambda \in (0, 1] \).

Assume that \( (\bar{y}, C, D) \in \partial \Omega \times \triangle \). Consider the case \( y_k = \{a, b\} \) for some \( k \in \{0, \ldots, n\} \). Suppose \( y_k = a \). From the boundary conditions, see that \( k \in \{1, \ldots, n - 1\} \). Thus, \( D^2y_{k+1} \geq 0 \). Now
\[
D^2y_{k+1} = \lambda m(x_k, y_k, Dy_k) < 0,
\]
a contradiction. The assumption \( y_k \neq b \) similarly leads to a contradiction.

See that \( \lambda m \) satisfies inequality (8), and thus, Lemma 1 is applicable to any solution \( \bar{y} \) to (10) satisfying \( \bar{a} \leq \bar{y} \leq \bar{b} \). Hence, \( \|Dy_k\| \leq N \) for \( k = 1, \ldots, n \) and \( \bar{y} \notin \partial \Omega \).

Argue by contradiction and assume that for \( (C, D) \in \partial \triangle \) we have
\[
G(C, D, \lambda y_d + (1 - \lambda)\phi_d(C)) = (0, 0), \tag{13}
\]
with \( \bar{a} \leq \bar{y} \leq \bar{b}, y_0 = \lambda C, \) and \( y_n = \lambda D \). See that \( \lambda y_d + (1 - \lambda)\phi_d(C) \) is a continuous function mapping into \([\alpha_d, \beta_d]\). Since \( G \) is strongly compatible, (13) cannot equal 0 for \( (C, D) \in \partial \triangle \). Thus,
\[
G(C, D, \lambda y_d + (1 - \lambda)\phi_d(C)) \neq (0, 0), \quad \text{for } (\bar{y}, C, D) \in \overline{\Omega} \times \partial \triangle.
\]
Thus, \( (\bar{y}, C, D) \notin \partial(\Omega \times \triangle) \) and the degree is defined on the bounded, open set \( \Omega \times \triangle \). By the invariance of the degree under homotopy (see [23]), we have
\[
d(\{(I - \lambda T)(\bar{y}), G(C, D, \lambda y_d + (1 - \lambda)\phi_d(C)), \Omega \times \triangle, 0\),
\]
\[
= d(\{(I - T)(\bar{y}), G(C, D, y_d)\), \Omega \times \triangle, 0\),
\]
\[
= d((I, G((C, D, \phi_d(C))\), \Omega \times \triangle, 0),
\]
\[
= d(G((C, D), \phi_d(C)), \triangle, 0) \neq 0,
\]
since \( 0 \in \Omega \) and \( G \) is strongly compatible. Thus, there is a solution \( (\bar{y}, C, D) \) as required. This concludes the proof.

The following is a generalisation of Theorem 2.5 in [16].
THEOREM 2. Let the assumptions of Theorem 1 hold. Given $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $0 < h < \delta$ and $\bar{y}$ is a solution of (3),(4), then there is a solution $y(x)$ of (1),(2) such that
\[ \max\{|y(x, \bar{y}) - y(x)| : 0 \leq x \leq 1\} \leq \varepsilon \]
and
\[ \max\{|v(x, \bar{y}) - y'(x)| : 0 \leq x \leq 1\} \leq \varepsilon, \]
where
\[ y(x, \bar{y}) = y_k + (x - x_k)y_k + (x - x_k)v_{k+1}, \quad \text{for} \ x_k \leq x \leq x_{k+1}, \]
and
\[ v(x, \bar{y}) = \begin{cases} D^2y_k + (x - x_k)^2y_{k+1}, & \text{for} \ x_k \leq x \leq x_{k+1}, \\ Dy_1, & \text{for} \ 0 \leq x \leq x_1. \end{cases} \]

The proof is similar to that of [16] and so is omitted.

REMARK 1. If solutions to the continuous problem (1),(2) are unique, it follows from Theorem 2 that solutions to (3),(4) converge to solutions of the continuous problem in the sense of Theorem 2.

LEMMA 5. Let $\alpha$, $\beta$, and $f$ satisfy the conditions of Theorem 1 and let the boundary conditions be given by
\[ G(y(0), y(1), y(a)) = (y(0) - y(a), y(1)) = (0, 0). \] (14)

If
\[ \alpha(0) < \alpha(a), \quad \beta(0) > \beta(a), \quad \alpha(1) < 0, \quad \text{and} \quad \beta(1) > 0, \] (15)
then there is a $\delta > 0$ such that for $0 < h < \delta$ the discrete analogue of problem (1),(14) has a solution $\bar{y}$ satisfying $\bar{\alpha} \leq \bar{y} \leq \bar{\beta}$.

PROOF. It needs to be shown that $G$ is strongly compatible with $\alpha$ and $\beta$. This follows from [13].

REMARK 2. It is not clear whether inequalities (15) are necessary for existence. They are “close” to necessary since the inequalities
\[ \alpha(0) \leq \alpha(a), \quad \beta(0) \geq \beta(a), \quad \alpha(1) \leq 0, \quad \text{and} \quad \beta(1) \geq 0 \]
are necessary and sufficient for the boundary conditions (14) to be compatible with $\alpha$, $\beta$, and hence, are necessary and sufficient for the associated continuous problem to have a solution (see, for example, [13]).

REMARK 3. Modifying arguments on pages 430 and 431 of [16], we can estimate $\delta > 0$ in Lemma 5 when $\alpha, \beta \in C^3([0, 1])$ and $f \in C^1([0, 1] \times \mathbb{R} \times \mathbb{R})$. It is not difficult to see that $\delta = \min_{1 \leq i \leq n} \delta_i$, where
\[ \delta_1 \leq \frac{\gamma}{4 \max |\beta''(x)|}, \]
\[ \delta_2 \leq \frac{\min \left\{ 1/2, \gamma \right\} \max |\partial_f/\partial_p|}{\max |\beta''(x)|}, \]
\[ \delta_3 \leq \frac{|\beta(0) - \beta(a)|}{\max |\beta'(x)|}, \]
and the estimates for $\delta_4$, $\delta_5$, and $\delta_6$ are obtained from the corresponding estimates for $\delta_1$, $\delta_2$, and $\delta_3$ by replacing $\beta$ with $\alpha$. The estimates can be improved by replacing $\max |\partial_f/\partial_p|$ by $\max_S |\partial_f/\partial_p|$ for suitable $S_i$, for $i = \alpha, \beta$ (see [16]).

There are other ways of bounding derivatives and difference quotients. In the continuous case, bounding surfaces have been used to obtain these a priori bounds (see [24,25]). The discrete analogue has been discussed in [17].
Remark 4. More general boundary conditions than (2) involve \( m \)-points and derivatives at the boundary. Once discrete \( m \)-point compatibility has been defined, similar existence results follow under conditions akin to those of Theorem 1.

Remark 5. Existence results when \( f \) is vector-valued also follow once the notion of upper and lower solutions is extended to systems, for example, as in [13].

REFERENCES