

ON TWO POINT BOUNDARY VALUE PROBLEMS FOR SECOND ORDER DIFFERENTIAL INCLUSIONS

LYNN ERBE¹, RUYUN MA², AND CHRISTOPHER C. TISDELL³

¹Department of Mathematics, The University of Nebraska - Lincoln
Lincoln, NE 68588 USA.
E-mail: lerbe@math.unl.edu

²Department of Mathematics, Northwest Normal University
Lanzhou 730070, P R China.
E-mail: mary@nwnu.ac.cn

³School of Mathematics, The University of New South Wales
Sydney 2052, Australia.
E-mail: cct@maths.unsw.edu.au

*Dedicated to John Baxley on the occasion of his
retirement from Wake Forest University.*

ABSTRACT. The authors consider a two point boundary value problem (BVP) for second order differential inclusions. Some inequalities are introduced so that all solutions to the BVP are bounded *a priori*. These bounds are then applied to prove the existence of solutions. The results are new even for differential equations with single-valued Carathéodory right-hand side.

AMS (MOS) Subject Classification. 34B15.

1. INTRODUCTION

In this paper, we consider the boundary value problem (BVP) for differential inclusions

$$y'' \in F(t, y, y'), \quad \text{for a.e. } t \in [0, T], \quad y(0) = A, \quad y(T) = B, \quad (1.1)$$

where $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathcal{K}(\mathbb{R}^n)$ is a multifunction, $\mathcal{K}(\mathbb{R}^n)$ is the family of all nonempty convex and compact subsets of \mathbb{R}^n , $0 < T \in \mathbb{R}$, and $A, B \in \mathbb{R}^n$, where “a.e.” stands for “almost every.” The interest in studying differential inclusions is twofold. First, the theory of differential inclusions enjoys applications to optimal control [1], [9], [11]. Secondly, results for differential inclusions can incorporate differential equations with discontinuities in the right hand side (or even when the right

hand side is inaccurately known) [1], [11]. This work is devoted to the existence of solutions to (1.1).

We introduce some new inequalities in the differential inclusion setting and prove some *a priori* bound results for all solutions to (1.1). The new inequalities are unrelated to the maximum principle methods of [5],[6] and [7]. The new *a priori* bound results are then applied, in conjunction with appropriate fixed point-based theorems, to prove the existence of solutions to (1.1). The results contained herein are new for Carathéodory multifunctions $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathcal{K}(\mathbb{R}^n)$, $n > 1$. However, the results are also new in the following special cases: when F is a scalar-valued Carathéodory multifunction; when F is a scalar- or vector-valued Carathéodory function; or, in all of the above cases, when F is independent of y' . For more on differential inclusions, we refer the reader to [1] - [11].

To present the new results we need to define the necessary notation.

Definition 1.1. Suppose that E and F are Banach spaces and $X \subset E$ and $Y \subset F$ are subsets. By $\mathcal{K}(Y)$ we denote the family of all nonempty convex and compact subsets of Y . A multivalued map $\Gamma : X \rightarrow \mathcal{K}(Y)$ is called upper semi-continuous (u.s.c.) if $\{x \in X : \Gamma(x) \subset U\}$ is an open subset of X for any open U in Y .

Definition 1.2. In what follows we will consider the Banach function spaces:

$$C([0, T]; \mathbb{R}^m) = \{u : [0, T] \rightarrow \mathbb{R}^m : u \text{ is continuous on } [0, T]\}$$

with the norm $\|u\|_\infty = \sup_{t \in [0, T]} \|u(t)\|$, where $\|\cdot\|$ denotes the usual Euclidean norm in \mathbb{R}^m ;

$$L^2([0, T]; \mathbb{R}^m) = \{u : [0, T] \rightarrow \mathbb{R}^m : \|u(t)\| \text{ is } L^2\text{-integrable}\}$$

with the norm

$$\|u\|_2 = \left(\int_a^b \|u(t)\|^2 dt \right)^{1/2};$$

$$H^k([0, T]; \mathbb{R}^m) = \{u : [0, T] \rightarrow \mathbb{R}^m : u \text{ has weak derivatives}$$

$$u^{(i)} \in L^2([0, T]; \mathbb{R}^m) \text{ for } 0 \leq i \leq k\}$$

with the norm

$$\|u\|_{2;k} = \max\{\|u^{(i)}\|_2 : 0 \leq i \leq k\}.$$

The spaces $H^k([0, T]; \mathbb{R}^m)$ are the usual Sobolev spaces of vector functions denoted also by $W^{k,2}([0, T]; \mathbb{R}^m)$ (for more details see [5]).

We now introduce the notion of a Carathéodory map or multifunction which will be central in the results to follow.

Definition 1.3. A multifunction $F : [0, T] \times \mathbb{R}^m \rightarrow \mathcal{K}(\mathbb{R}^n)$ is said to be a *Carathéodory multifunction* if it satisfies the conditions:

- (i) the map $t \rightarrow F(t, u)$ is Lebesgue measurable for each $u \in \mathbb{R}^m$;
- (ii) the map $u \rightarrow F(t, u)$ is u.s.c. for each $t \in [0, T]$;
- (iii) for any $r \geq 0$ there is a function $\psi_r \in L^2[0, T]$ such that for all $t \in [0, T]$, $u \in \mathbb{R}^m$ with $\|u\| \leq r$ and $y \in F(t, u)$, we have $\|y\| \leq \psi_r(t)$.

The following general existence theorem will be very useful in the remainder of the paper.

Theorem 1.4. ([7, Theorem 3.1]) *Suppose that $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathcal{K}(\mathbb{R}^n)$ is a Carathéodory multifunction. If there exists a positive constant M such that:*

$$\max\{\|y\|_\infty, \|y'\|_\infty\} < M,$$

for all solutions y to

$$y'' \in \lambda F(t, y, y'), \quad \text{for a.e. } t \in [0, T], \quad y(0) = \lambda A, \quad y(T) = \lambda B, \quad (1.2)$$

for $\lambda \in [0, 1]$, then (1.1) has at least one solution in $H^2([0, T]; \mathbb{R}^n)$.

2. MAIN RESULTS

The main question that Theorem 1.4 raises is this: How can we ensure that all solutions to the family of BVPs (1.2) are bounded *a priori*? The following results introduce some new inequalities that will guarantee the desired bounds. Set

$$\|F(t, y, y')\|_* = \sup\{\|w\| : w \in F(t, y, y')\}.$$

Lemma 2.1. *Let $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathcal{K}(\mathbb{R}^n)$ be a Carathéodory multifunction. If there exists non-negative constants α_1 and K_1 such that*

$$\|F(t, y, p)\|_* \leq \operatorname{ess\,inf}_{t \in [0, T]} \inf\{2\alpha_1 [y \cdot w + \|p\|^2] + K_1 : (y, p) \in \mathbb{R}^{2n} \text{ and } w \in F(t, y, p)\}, \quad (2.1)$$

then all solutions y to (1.1) satisfy

$$\|y\|_\infty \leq \alpha_1 \beta^2 + \beta + K_1 T^2 / 8 := R, \quad (2.2)$$

where $\beta := \max\{\|A\|, \|B\|\}$.

Proof. The inclusion (1.1) is equivalent to

$$y(t) \in - \int_0^T G(t, s) F(s, y(s), y'(s)) ds + \phi(t), \quad \text{for a.e. } t \in [0, T], \quad (2.3)$$

where

$$0 \leq G(t, s) := \begin{cases} \frac{(T-t)s}{T}, & 0 \leq s \leq t \leq T, \\ \frac{t(T-s)}{T}, & 0 \leq t \leq s \leq T, \end{cases}$$

and

$$\phi(t) := \frac{TA + (B - A)t}{T}, \quad \text{for } t \in [0, T].$$

Taking norms in (2.3), we obtain

$$\begin{aligned} \|y(t)\| &\leq \int_0^T G(t, s) \|F(s, y(s), y'(s))\| ds + \beta, \\ &\leq \int_0^T G(t, s) \operatorname{ess\,inf}_{[0, T]} \inf \{ [2\alpha_1(y(s) \cdot w(s) + \|y'(s)\|^2) + K_1] \} ds + \beta, \\ &\leq \int_0^T G(t, s) [2\alpha_1(y(s) \cdot y''(s) + \|y'(s)\|^2) + K_1] ds + \beta, \\ &= \int_0^T G(t, s) [\alpha_1 r''(s) + K_1] ds + \beta, \quad \text{where } r(t) = \|y(t)\|^2, \\ &= \int_0^t \left(\frac{(T-t)s}{T} \right) \alpha_1 r''(s) ds + \int_t^T \left(\frac{t(T-s)}{T} \right) \alpha_1 r''(s) ds \\ &\quad + K_1 \int_0^T G(t, s) ds + \beta. \end{aligned}$$

Integrating by parts and using the fact that

$$\int_0^T G(t, s) ds \leq T^2/8,$$

we obtain

$$\begin{aligned} \|y(t)\| &\leq \alpha_1 \left[\left(\frac{T-t}{T} \right) \|y(0)\|^2 + \frac{t}{T} \|y(T)\|^2 \right] + K_1 T^2/8 + \beta, \\ &\leq \alpha_1 \left[\left(\frac{T-t}{T} \right) \beta^2 + \frac{t}{T} \beta^2 \right] + K_1 T^2/8 + \beta, \\ &= \alpha_1 \beta^2 + \beta + K_1 T^2/8, \end{aligned}$$

and the result follows. This concludes the proof. \square

For scalar Carathéodory multifunctions ($n = 1$), we have the following new result as a corollary to Lemma 2.1.

Corollary 2.2. *Let $F : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{K}(\mathbb{R})$ be a Carathéodory multifunction. If there exist non-negative constants α_1 and K_1 such that*

$$|F(t, y, y')|_* \leq \operatorname{ess\,inf}_{t \in [0, T]} \inf \{ 2\alpha_1 [yw + (y')^2] + K_1 : (y, y') \in \mathbb{R}^2, w \in F(t, y, y') \} \quad (2.4)$$

then all solutions y to (1.1) (with $n = 1$) satisfy (2.2), where $\beta := \max\{|A|, |B|\}$.

If F is independent of y' , then (1.1) reduces to

$$y'' \in F(t, y), \quad \text{for a.e. } t \in [0, T], \quad y(0) = A, y(T) = B, \quad (2.5)$$

and Lemma 2.1 and Corollary 2.2 reduce to the two following new results.

Lemma 2.3. *Let $F : [0, T] \times \mathbb{R}^n \rightarrow \mathcal{K}(\mathbb{R}^n)$ be a Carathéodory multifunction. If there exist non-negative constants α_1 and K_1 such that*

$$\|F(t, y)\|_* \leq \operatorname{ess\,inf}_{t \in [0, T]} \inf \{2\alpha_1 y \cdot w + K_1 : y \in \mathbb{R}^n, w \in F(t, y)\}, \quad (2.6)$$

then all solutions to (2.5) satisfy (2.2).

Proof. Notice that

$$2\alpha_1 y \cdot w + K_1 \leq 2\alpha_1 [y \cdot w + \|y'\|^2] + K_1 \leq \alpha_1 r'' + K_1,$$

where $r = \|y\|^2$. The remaining argument is identical to that of Lemma 2.1. \square

Corollary 2.4. *Let $F : [0, T] \times \mathbb{R} \rightarrow \mathcal{K}(\mathbb{R})$ be a Carathéodory multifunction. If there exist non-negative constants α_1 and K_1 such that*

$$|F(t, y)|_* \leq \operatorname{ess\,inf}_{t \in [0, T]} \inf \{2\alpha_1 y w + K_1 : y \in \mathbb{R}, w \in F(t, y)\}, \quad (2.7)$$

then all solutions to (2.5) satisfy (2.2) where $\beta = \max\{|A|, |B|\}$.

To ensure *a priori* bounds on first derivatives of solutions to (1.2), we will need the following well known Nagumo condition [7].

Lemma 2.5. *Let $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathcal{K}(\mathbb{R}^n)$ be a Carathéodory multifunction. If there exists $\phi : [0, \infty) \rightarrow (0, \infty)$ and non-negative constants α_2 , K_2 , and R such that*

$$\frac{s}{\phi(s)} \in L_{loc}^\infty [0, \infty), \quad \int_0^\infty \frac{s}{\phi(s)} ds = \infty, \quad (2.8)$$

$$\|F(t, y, y')\|_* \leq \phi(\|y'\|), \quad \text{for a.e. } t \in [0, T] \text{ and all } \|y\| \leq R, y' \in \mathbb{R}^n, \quad (2.9)$$

$$\|F(t, y, y')\|_* \leq 2\alpha_2 [y \cdot w + \|y'\|^2] + K_2,$$

$$\text{for a.e. } t \in [0, T] \text{ and all } \|y\| \leq R, y' \in \mathbb{R}^n, w \in F(t, y, y'), \quad (2.10)$$

then for all solutions to (1.1) satisfying $\|y\|_\infty \leq R$, we also have $\|y'\|_\infty \leq N$ for some constant N .

Lemma 2.5 may be simplified, if R or α_2 is small enough, to give the following result.

Lemma 2.6. *Let $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathcal{K}(\mathbb{R}^n)$ be a Carathéodory multifunction. If there exist non-negative constants α_2, K_2, R such that (2.10) holds with $2\alpha_2 R < 1$, then for all solutions to (1.1) satisfying $\|y\|_\infty \leq R$, we also have $\|y'\|_\infty \leq N$ for some constant N .*

Proof. If (2.10) holds, then taking norms on both sides of (2.10) and rearranging the terms, we obtain

$$\|F(t, y, y')\| \leq \frac{\|y'\|^2 + K_2}{1 - 2\alpha_2 R}, \quad \text{for a.e. } t \in [0, T] \text{ and all } \|y\| \leq R, y' \in \mathbb{R}^n.$$

So the conditions of Lemma 2.5 will hold with

$$\phi(s) = \frac{s^2 + K_2}{1 - 2\alpha_2 R},$$

and the result then follows from Lemma 2.3. \square

Remark 2.7. If $F : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{K}(\mathbb{R})$ and F satisfies the conditions of Lemma 2.5 with (2.10) removed, then the conclusion of Lemma 2.5 will still hold.

Remark 2.8. If F is independent of y' , then *a priori* bounds on y' follow naturally, once an *a priori* bound on y is obtained, say from Lemma 2.3 or Corollary 2.4. To verify this, just differentiate in (2.3) and then take norms.

We are now in a position to present some existence theorems for solutions to (1.1).

Theorem 2.9. *Let $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathcal{K}(\mathbb{R}^n)$ be a Carathéodory multifunction. If there exists $\phi : [0, \infty) \rightarrow (0, \infty)$ and non-negative constants $\alpha_1, K_1, \alpha_2, K_2$, and R such that (2.1), (2.8), (2.9), and (2.10) hold, then (1.1) has at least one solution in $H^2([0, T]; \mathbb{R}^n)$.*

Proof. In view of Theorem 1.4, we will show that the inequalities (2.1), (2.8), (2.9), and (2.10) imply that Lemmas 2.1 and 2.5 are applicable to the family of BVPs (1.2). Thus, all solutions to (1.2) will be bounded *a priori* and the existence of solutions to (1.1) will follow from Theorem 1.4.

If (2.1) holds, then multiplying both sides of (2.1) by $\lambda \in [0, 1]$, we obtain

$$\begin{aligned} \|\lambda F(t, y, y')\|_* &\leq 2\alpha_1 [y \cdot \lambda w + \lambda \|y'\|^2] + \lambda K_1, \\ &\leq 2\alpha_1 [y \cdot w_1 + \|y'\|^2] + K_1, \\ &\quad \text{for a.e. } t \in [0, T] \text{ and all } (y, y') \in \mathbb{R}^{2n}, w_1 \in \lambda F(t, y, y'). \end{aligned}$$

Therefore, Lemma 2.1 applies to (1.2), and so $\|y\|_\infty \leq R$ for all solutions to (1.2).

If (2.9) holds, then

$$\|\lambda F(t, y, y')\|_* \leq \|F(t, y, y')\|_* \leq \phi(\|y'\|), \quad \text{for a.e. } t \in [0, T] \text{ and all } \|y\| \leq R, y' \in \mathbb{R}^n.$$

Inequality (2.10) will hold with F replaced by λF and w replaced by w_1 as per the initial argument of this proof. Therefore, Lemma 2.5 is applicable to (1.2), and hence all solutions to (1.2) satisfying $\|y\|_\infty \leq R$ also satisfy $\|y'\|_\infty \leq N$ for some $N \geq 0$.

Finally, choose $M := \max\{R, N\} + 1$; then all solutions to (1.2) satisfy

$$\max\{\|y\|_\infty, \|y'\|_\infty\} < M, \quad \text{for all } \lambda \in [0, 1].$$

By Theorem 1.4, the BVP (1.1) has at least one solution in $H^2([0, T]; \mathbb{R}^n)$. \square

Corollary 2.10. *Let $F : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{K}(\mathbb{R})$ be a Carathéodory multifunction. If there exists $\phi : [0, \infty) \rightarrow (0, \infty)$ and non-negative constants α_1 , K_1 , and R such that (2.4), (2.8), and (2.9) hold, then (for $n = 1$) (1.1) has at least one solution in $H^2([0, T]; \mathbb{R})$.*

Corollary 2.11. *Let $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathcal{K}(\mathbb{R}^n)$ be a Carathéodory multifunction. If there exist non-negative constants α_1 , K_1 , α_2 , K_2 , and R such that (2.1), (2.10), and $2\alpha_2 R < 1$ hold, then (1.1) has at least one solution in $H^2([0, T]; \mathbb{R}^n)$.*

If the constants in Theorem 2.9 or Corollaries 2.10 or 2.11 satisfy $\alpha_1 = \alpha_2$ and $K_1 = K_2$, then the inequalities (2.1) and (2.10) can be combined to give the following result.

Corollary 2.12. *Let $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathcal{K}(\mathbb{R}^n)$ be a Carathéodory multifunction. If there exist non-negative constants α and K such that*

$$\|F(t, y, y')\|_* \leq \text{ess inf}_{t \in [0, T]} \inf\{2\alpha [y \cdot w + \|y'\|^2] + K : y \in \mathbb{R}^n, w \in F(t, y, y')\},$$

and either

- (i) *there exists $\phi : [0, \infty) \rightarrow (0, \infty)$ such that (2.8) and (2.9) hold, or*
- (ii) $2\alpha R < 1$,

then (1.1) has at least one solution in $H^2([0, T]; \mathbb{R}^n)$.

When considering (2.5), Theorem 2.9 reduces to the following new result.

Corollary 2.13. *Let $F : [0, T] \times \mathbb{R}^n \rightarrow \mathcal{K}(\mathbb{R}^n)$ be a Carathéodory multifunction. If there exist non-negative constants α_1 , K_1 , and R such that (2.6) holds, then (2.5) has at least one solution in $H^2([0, T]; \mathbb{R}^n)$.*

If F is scalar-valued, then Corollary 2.13 becomes the following new result.

Corollary 2.14. *Let $F : [0, T] \times \mathbb{R} \rightarrow \mathcal{K}(\mathbb{R})$ be a Carathéodory multifunction. If there exist non-negative constants α_1 , K_1 , and R such that (2.7) holds, then (for $n = 1$) (1.1) has at least one solution in $H^2([0, T]; \mathbb{R})$.*

3. SINGLE-VALUED CARATHÉODORY F

For Carathéodory single-valued functions, the problem (1.1) reduces to

$$y'' = f(t, y, y'), \quad \text{for a.e. } t \in [0, T], \quad y(0) = A, \quad y(T) = B, \quad (3.1)$$

and the following new results follow as corollaries to Theorem 2.9. The proofs involve applying the earlier theorems to (1.1) for the special case $F(t, y, y') = \{f(t, y, y')\}$.

Corollary 3.1. *Let $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a Carathéodory function. If there exists $\phi : [0, \infty) \rightarrow (0, \infty)$ and non-negative constants $\alpha_1, K_1, \alpha_2, K_2$, and R such that*

$$\begin{aligned} \|f(t, y, y')\| &\leq 2\alpha_1 [y \cdot f(t, y, y') + \|y'\|^2] + K_1, \\ &\text{for a.e. } t \in [0, T] \text{ and all } (y, y') \in \mathbb{R}^{2n}, \end{aligned}$$

(2.8) and (2.9) hold, and

$$\begin{aligned} \|f(t, y, y')\| &\leq 2\alpha_2 [y \cdot f(t, y, y') + \|y'\|^2] + K_2, \\ &\text{for a.e. } t \in [0, T] \text{ and all } \|y\| \leq R, \quad y' \in \mathbb{R}^n, \end{aligned} \quad (3.2)$$

then (3.1) has at least one solution in $H^2([0, T]; \mathbb{R}^n)$.

Corollary 3.2. *Let $f : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function. If there exists $\phi : [0, \infty) \rightarrow (0, \infty)$ and non-negative constants $\alpha_1, K_1, \alpha_2, K_2$, and R such that*

$$\begin{aligned} |f(t, y, y')| &\leq 2\alpha_1 [yf(t, y, y') + (y')^2] + K_1, \\ &\text{for a.e. } t \in [0, T] \text{ and all } (y, y') \in \mathbb{R}^2, \end{aligned}$$

(2.8) and (2.9) hold, then (for $n = 1$) (3.1) has at least one solution in $H^2([0, T]; \mathbb{R})$.

Corollary 3.3. *Let the conditions of Corollary 3.1 hold, with the assumptions on ϕ and (3.1) removed. If $2\alpha_2 R < 1$, then (3.1) has at least one solution in $H^2([0, T]; \mathbb{R})$.*

Corollary 3.4. *Let $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathcal{K}(\mathbb{R}^n)$ be a Carathéodory function. If there exist non-negative constants α, K , and R such that*

$$\begin{aligned} \|f(t, y, y')\| &\leq 2\alpha [y \cdot f(t, y, y') + \|y'\|^2] + K, \\ &\text{for a.e. } t \in [0, T] \text{ and all } (y, y') \in \mathbb{R}^n, \end{aligned}$$

with $2\alpha R < 1$, then (3.1) has at least one solution in $H^2([0, T]; \mathbb{R}^n)$.

When f is Carathéodory and independent of y' , the following new results are obtained from Corollaries 3.1 and 3.2.

Corollary 3.5. *Let $f : [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a Carathéodory function. If there exist non-negative constants α_1 and K_1 such that*

$$\|f(t, y, y')\| \leq 2\alpha_1 y \cdot f(t, y) + K_1, \quad \text{for a.e. } t \in [0, T] \text{ and all } y \in \mathbb{R}^n,$$

then the BVP

$$y'' = f(t, y), \quad \text{for a.e. } t \in [0, T], \quad y(0) = A, \quad y(T) = B, \quad (3.3)$$

has at least one solution in $H^2([0, T]; \mathbb{R}^n)$.

Corollary 3.6. *Let $f : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function. If there exist non-negative constants α_1 and K_1 such that*

$$|f(t, y)| \leq 2\alpha_1 y f(t, y) + K_1, \quad \text{for a.e. } t \in [0, T] \text{ and all } y \in \mathbb{R},$$

then (for $n = 1$) (3.3) has at least one solution in $H^2([0, T]; \mathbb{R})$.

4. Examples

Some examples are now presented to highlight the previous results.

Example 4.1. Consider the Carathéodory multifunction $F : [0, T] \times \mathbb{R} \rightarrow \mathcal{K}(\mathbb{R})$ given by

$$F(t, y) = \cup_{\varepsilon \in [0, 1]} t(1 + \varepsilon)y^{2n+1}, \quad n \text{ is any natural number.} \quad (4.1)$$

For F in (4.1), we claim that (2.5) has at least one solution in $H^2([0, T]; \mathbb{R})$.

In view of Corollary 2.14, we need to show that (2.7) holds for some non-negative constants α_1 and K_1 . We see that for each $\varepsilon \in [0, 1]$, $t \in [0, T]$, and $y \in \mathbb{R}$, we have

$$t(1 + \varepsilon)|y^{2n+1}| \leq t(1 + \varepsilon)(y^{2n+2} + 1)$$

since $k = 2n + 1$ is odd. Thus, for every $f(t, y) \in F(t, y)$, we have

$$\begin{aligned} |f(t, y)| &= t(1 + \varepsilon)|y^{2n+1}| \leq t(1 + \varepsilon)(y^{2n+2} + 1), \\ &= yt(1 + \varepsilon)y^{2n+1} + t(1 + \varepsilon) = yf(t, y) + t(1 + \varepsilon), \\ &\leq 2\alpha_1 yf(t, y) + K_1, \end{aligned}$$

for the choices $\alpha_1 = 1/2$ and $K_1 = 2T$. Hence, (2.7) holds, and by Corollary 2.14, the BVP has a solution.

Example 4.2. Consider the Carathéodory multifunction $F : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{K}(\mathbb{R})$ given by

$$F(t, y, y') = \cup_{\varepsilon \in [0, 1]} (t + \varepsilon) \sin(y) \sin(y'). \quad (4.2)$$

For F in (4.2), we claim that (1.1) has at least one solution in $H^2([0, T]; \mathbb{R})$. In view of Corollary 2.10, we need to show that (2.4) holds for some non-negative constants α_1 and K_1 .

Now for each $\varepsilon \in [0, 1]$, $t \in [0, T]$, $(y, y') \in \mathbb{R}^2$, and $f(t, y, y') \in F(t, y, y')$, we have

$$|f(t, y, y')| = (t + \varepsilon)|\sin(y) \sin(y')| \leq T + 1 = \alpha_1 yf(t, y, y') + K_1,$$

for the choices $\alpha_1 = 0$ and $K_1 = T + 1$. Hence, (2.4) holds. Also, the function $\phi : [0, \infty) \rightarrow (0, \infty)$ given by the constant function $\phi = T + 1$ satisfies (2.8) and (2.9). Thus, by Corollary 2.10, the BVP has a solution.

The maximum principle results in [5], [6] and [7] do not apply to the previous example.

ACKNOWLEDGMENTS

C. C. Tisdell gratefully acknowledges the financial support of the Australian Research Council's Discovery Projects (DP0450752).

REFERENCES

- [1] J. P. Aubin and A. Cellina, *Differential inclusions. Set-valued maps and viability theory*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Vol. 264, Springer-Verlag, Berlin, 1984.
- [2] E. P. Avgerinos, N. S. Papageorgiou and N. Yannakakis, Periodic solutions for second order differential inclusions with nonconvex and unbounded multifunction, *Acta Math. Hungar.* 83 (1999), 303–314.
- [3] R. Bader, B. D. Gel'fan, M. Kamenskii and V. Obukhovskii, On the topological dimension of the solutions sets for some classes of operator and differential inclusions, *Discuss. Math. Differ. Incl. Control Optim.* 22 (2002), 17–32.
- [4] A. Boucherif and B. Chanane, Boundary value problems for second order differential inclusions, *Int. J. Differ. Equ. Appl.* 7 (2003), 147–151.
- [5] L. H. Erbe and W. Krawcewicz, Boundary value problems for second order nonlinear differential inclusions, in: *Qualitative Theory of Differential Equations (Szeged, 1988)*, pp. 163–171, Colloq. Math. Soc. Janos Bolyai, 53, North-Holland, Amsterdam, 1990.
- [6] L. H. Erbe and W. Krawcewicz, Boundary value problems for differential inclusions, in: *Differential Equations (Colorado Springs, CO, 1989)*, 115–135, Lecture Notes in Pure and Appl. Math., 127, Dekker, New York, 1991.
- [7] L. H. Erbe and W. Krawcewicz, Nonlinear boundary value problems for differential inclusions $y'' \in F(t, y, y')$, *Ann. Polon. Math.* 54 (1991), 195–226.
- [8] D. A. Kandilakis and N. S. Papageorgiou, Existence theorems for nonlinear boundary value problems for second order differential inclusions, *J. Differential Equations* 132 (1996), 107–125.
- [9] M. Kisielewicz, *Differential Inclusions and Optimal Control*, Mathematics and its Applications (East European Series), Vol. 44, Kluwer Academic Publishers Group, Dordrecht; PWN—Polish Scientific Publishers, Warsaw, 1991.
- [10] T. Pruszko, Some applications of the topological degree theory to multivalued boundary value problems, *Dissertationes Math. (Rozprawy Mat.)* 229 (1984).
- [11] G. V. Smirnov, *Introduction to the Theory of Differential Inclusions*, Graduate Studies in Mathematics, Vol. 41, American Mathematical Society, Providence, 2002.