

**MATH 2019
ENGINEERING MATHEMATICS 2CE.**

**A Revision of ODEs:
Ordinary Differential Equations.**

The objective of this handout is to briefly revise the solution methods to “2nd-order” ODEs which were originally taught in MATH1231.

Many problems in physics and engineering involve the concept of a rate of change and it is in such problems that differential equations may appear.

The **order** of a differential equation is the highest derivative that appears in the equation. Since only *ordinary* derivatives, rather than partial derivatives, are involved, the equations are called *ODEs*.

A *solution* to an differential equation is a function which is suitably differentiable and which satisfies the given equation.

Second Order Linear Differential Equations with Constant Co-efficients:

The second order linear ODE with constant co-efficients, has the form

$$y'' + ay' + by = f(x)$$

where y is a function of x which we have to determine. If $f(x) = 0$, then we say that the equation

$$y'' + ay' + by = 0 \quad (*)$$

is *homogeneous*. Note carefully that we are using this word in a **different** sense to that which we used it for in regard to first order equations. We will deal firstly with the homogeneous equation.

Firstly observe that if we have two solutions $y_1(x)$ and $y_2(x)$ to (*), then any linear combination of these is also a solution, since

Now in order to solve the homogeneous equations $y'' + ay' + by = 0$, we seek a function y which does not change a great deal when differentiated. We try therefore $y = Ae^{\lambda x}$, where A is a non-zero constant.

Ex. Solve $y'' - 5y' + 6y = 0$. Try $y = Ae^{\lambda x}$.
Substitution yields

The quadratic is called the

characteristic equation

associated with the ODE. We normally go straight to the characteristic equation, solve it and simply write down the general solution.

In general the characteristic equation for

$$ay'' + by' + cy = 0 \text{ is } a\lambda^2 + b\lambda + c = 0.$$

Ex: Solve the initial value problem

$$y'' - 4y' - 5y = 0, \text{ with } y(0) = 2, y'(0) = 10.$$

Repeated Roots:

Ex. Solve

$$y'' - 4y' + 4y = 0.$$

The char. eqn. has repeated root, $\lambda = 2$, so a solution is

$$y_1 = Ae^{2x}.$$

We now try the following trick. We look for a solution of the form

$$y_2 = v(x)y_1 = v(x)e^{2x}.$$

Substitution leads us to $e^{2x}v''(x) = 0$ and so $v(x) = Bx$, thus $y_2(x) = Bxe^{2x}$.

The sort of technique we used above works in general and so if the characteristic equation has a single repeated root, λ , then the general solution is

$$y = Ae^{\lambda x} + Bxe^{\lambda x}.$$

Ex. Solve

$$y'' + 6y' + 9y = 0$$

subject to $y(0) = 0$, $y'(0) = 5$.

Complex Roots:

Finally, the third possibility is that the roots are complex, and we know that complex roots of a real quadratic occur in pairs as

$$\lambda = \alpha \pm i\beta.$$

Thus, as above we have solution

$$\begin{aligned} y &= C e^{(\alpha+i\beta)x} + D e^{(\alpha-i\beta)x} \\ &= (C + D) e^{\alpha x} \cos \beta x + i(C - D) e^{\alpha x} \sin \beta x. \end{aligned}$$

(By Euler's formula $e^{i\theta} = \cos \theta + i \sin \theta$.)

Now the constants C and D are arbitrary, so to get real solutions, we choose C, D such that $A = C + D$ and $B = i(C - D)$ are both real and so the general real solution is

$$y = e^{\alpha x} (A \cos \beta x + B \sin \beta x).$$

Ex. Solve

$$y'' - 2y' + 5y = 0.$$

The characteristic equation has solution

To summarise, we solve the characteristic equation of

$$y'' + ay' + by = 0$$

which is

$$\lambda^2 + a\lambda + b = 0.$$

If

- roots are real and distinct, λ_1, λ_2 , the general solution is

- roots are equal, λ , the general solution is

- roots are complex, $\lambda = \alpha \pm i\beta$, the general solution is

The Non-Homogeneous Case:

Ex. Solve

$$y'' - 5y' + 6y = 2x + 3.$$

Ex: Solve

$$y'' - 5y' + 6y = 12e^{5x}.$$

We can construct a table of what to try as a particular solution for given $f(x)$.

| |
|--|
| $f(x)$ |
| $P(x)$, a n th deg. polyn. |
| $P(x)e^{ax}$ |
| $P(x) \cos ax$ |
| $P(x) \sin ax$ |
| $P(x)e^{ax} \sin bx$ or $P(x)e^{ax} \cos bx$ |

Respective Guess for y_p .

$Q(x)$, n th deg. polyn.

$Q(x)e^{ax}$

$Q_1(x) \cos ax + Q_2(x) \sin ax$

$Q_1(x) \cos ax + Q_2(x) \sin ax$

$Q_1(x)e^{ax} \cos bx + Q_2(x)e^{ax} \sin bx$

***Care must be taken however, when using the above table as the following examples will show. ***

Ex. Solve $y'' - 5y' + 6y = 12e^{2x}$.

An even more unpleasant example is:

Ex. Solve $y'' - 4y' + 4y = 2e^{2x}$.

Thus, as a general rule, if the right hand side of the equation has a function which is already in the kernel (i.e. one of the homogeneous solutions), we multiply by x until the resulting function is no longer a solution to the homogeneous equation.

Ex. Solve $y'' + y = \cos x$, such that $y(0) = 3, y'(0) = 0$