MATH5215, Solutions to Some Questions Involving Σ

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Throughout we assume $\Delta C(t) = 0$.

1. (a) We use the identity

$$\sin u - \sin v = 2\sin\left[\frac{1}{2}(u-v)\right]\cos\left[\frac{1}{2}(u+v)\right].$$

$$\begin{aligned}\Delta \sin a(t-1/2) &= \sin a(t+1/2) - \sin a(t-1/2)\\ &= 2\sin\left[\frac{a}{2}((t+1/2)-(t-1/2))\right]\cos\left[\frac{a}{2}((t+1/2)+t-1/2)\right]\\ &= 2\sin\frac{a}{2}\cos at.\end{aligned}$$

So we have

$$\Delta\left(\frac{\sin a(t-1/2)}{2\sin(a/2)}\right) = \cos at$$

thus

$$\sum \cos at = \frac{\sin a(t-1/2)}{2\sin(a/2)} + C(t), \quad a \neq 2n\pi$$

(b) The symmetry identity (Question Sheet 1) gives

$$\Delta_t \binom{r+t}{t-1} = \Delta_t \binom{r+t}{t+r-(t-1)} = \Delta_t \binom{r+t}{r+1} = \binom{r+t}{r}$$

so

$$\sum \binom{t+r}{r} = \binom{t+r}{t-1} + C(t)$$

2. We use the summation by parts formula

$$\sum (y(t)\Delta z(t)) = y(t)z(t) - \sum \Delta y(t) Ez(t).$$

(a) For $\sum t \sin t$ let $y(t) = t$ and let $\Delta z(t) = \sin t$. So

$$\Delta y(t) = 1, \quad z(t) = \sum \sin t = -\frac{\cos(t-1/2)}{2\sin 1/2}$$

$$Ez(t) = z(t+1) = -\frac{\cos(t+1/2)}{2\sin 1/2}.$$

Using the summation by parts formula

$$\begin{aligned} \sum t\sin t &= -t\frac{\cos(t-1/2)}{2\sin 1/2} - \sum -\frac{\cos(t+1/2)}{2\sin 1/2} \\ &= -t\frac{\cos(t-1/2)}{2\sin 1/2} + \frac{1}{2\sin 1/2}\sum \cos(t+1/2) \\ &= -t\frac{\cos(t-1/2)}{2\sin 1/2} + \frac{1}{2\sin 1/2}\frac{\sin t}{2\sin 1/2} + C(t) \end{aligned}$$

where, in the final line, we used the facts that

$$\Delta \sin t = 2\sin\frac{1}{2}\cos(t+1/2), \text{ so } \sin t = 2\sin\frac{1}{2}\sum\cos(t+1/2) + C(t)$$

and thus

$$\sum \cos(t+1/2) = \frac{\sin t}{2\sin\frac{1}{2}}.$$

(b) For $\sum t^2 3^t$ let $y(t) = t^2$ and let $\Delta z(t) = 3^t$. So

$$\Delta y(t) = 2t+1, \quad z(t) = \sum 3^t = \frac{3^t}{2}$$

$$Ez(t) = z(t+1) = \frac{3}{2}3^t.$$

Using the summation by parts formula (twice)

$$\begin{aligned}\sum t^2 3^t &= t^2\frac{3^t}{2} - \frac{3}{2}\sum -(2t+1)3^t \\ &= t^2\frac{3^t}{2} - \left(\sum 3t3^t\right) - \frac{3}{2}\frac{3^t}{2}, \qquad \text{now let } y(t)=3t,\ z(t)=3^t/2, \\ &= t^2\frac{3^t}{2} - \frac{3}{2}\frac{3^t}{2} - 3t\frac{3^t}{2} + \frac{9}{2}\sum 3^t \\ &= \frac{3^t}{2}\left[t^2 - 3t + 3\right] + C(t).\end{aligned}$$

(c) Answer:

$$\sum \frac{t}{(t+1)(t+2)(t+3)} = -\frac{1}{2}t\left(t^{\underline{-2}}\right) - \frac{1}{2}(t+1)^{\underline{-1}} + C(t),$$

(d)

$$\sum \binom{t}{2}\binom{t}{7} = \binom{t}{2}\binom{t}{7} - t\binom{t+1}{9} + \binom{t+2}{10} + C(t)$$

(e)

$$\sum \binom{t}{2}^2 = \sum \binom{t}{2}\binom{t}{2} = \binom{t}{2}\binom{t}{3} - t\binom{t+1}{4} + \binom{t+2}{5} + C(t)$$

3. Since z_n is an indefinite sum (anti-difference) of y_n we have $\Delta z_n = y_n$. Thus

$$\begin{aligned}&\sum_{k=m}^{n-1} y_k = \sum_{k=m}^{n-1} \Delta z_k = \sum_{k=m}^{n-1}(z_{k+1} - z_k) \\ &= (z_{m+1} - z_m) + (z_{m+2} - z_{m+1}) + \cdots + (z_{n-1} - z_{n-2}) + (z_n - z_{n-1}) \\ &= z_n - z_m\end{aligned}$$

4. From tables we know that

$$\sum \cos ak = \frac{\sin a(k-1/2)}{2\sin(a/2)} + C(t)$$

so

$$\sum_{k=1}^{n-1} \cos ak = \left[\frac{\sin a(k-1/2)}{2\sin(a/2)}\right]_{k=1}^{k=n}$$

and the identity follows.

5. Left as an exercise

6. Here we notice

$$\sum_{k=1}^{n-1} \frac{k}{2^k} = \sum_{k=1}^{n-1} \left(\frac{1}{2}\right)^k k$$

and use summation by parts. Let $y(k) = k$ and $\Delta z(k) = (1/2)^k$. So

$$\Delta y(k) = 1, \quad z(k) = -2\left(\frac{1}{2}\right)^k, \quad Ez(k) = -\left(\frac{1}{2}\right)^k.$$

Hence

$$\begin{aligned}\sum_{k=1}^{n-1} \frac{k}{2^k} &= \left[-2k\left(\frac{1}{2}\right)^k\right]_{k=1}^{k=n} + \sum_{k=1}^{n-1}\left(\frac{1}{2}\right)^k \\ &= -2n\left(\frac{1}{2}\right)^n + 1 + \left[-2\left(\frac{1}{2}\right)^k\right]_{k=1}^{n} \\ &= -2n\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{2}\right)^n + 2\end{aligned}$$

7. Notice that

$$\sum_{k=2}^{n-1} k^2(k-1) = \sum_{k=2}^{n-1} k\,[k(k-1)] = \sum_{k=2}^{n-1} k\left[k^{\underline{2}}\right]$$

and use summation by parts.

8. Left as an exercise.