NAME OF CANDIDATE: . . . .
STUDENT NUMBER: . . . .

THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS

June 2005
PRACTICE EXAM

MATH5215
Dynamic Equations on Time Scales

TIME ALLOWED – 2 hours

TOTAL NUMBER OF QUESTIONS – 5

ANSWER ALL QUESTIONS

THE QUESTIONS ARE OF EQUAL VALUE

ANSWER EACH QUESTION IN A SEPARATE BOOK

THIS PAPER MAY **NOT** BE RETAINED BY THE CANDIDATE

ONLY THE PROVIDED ELECTRONIC CALCULATORS MAY BE USED

If you are ill during this examination and wish to request special consideration, you must inform the invigilator that you are ill, and ensure that the invigilator makes a note of this fact on your examination booklets. These examination booklets will not be marked, and you must apply for special consideration, through the Student Centre, within 3 days of this examination.

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

Answer each question in a separate book
- This is a PRACTICE EXAM

1. a) Solve the difference equations

i)
$$y(t+1) - y(t) = 3^t + t^{\underline{3}},$$

ii)
$$y(t+2) - 2y(t+1) + y(t) = t^{\underline{n}}, \quad n \in \mathbb{N}.$$

b) Find the solution to the difference equation
$$y(t+1) - ty(t) = -3^t.$$

c) i) Prove the following summation by parts identity for definite sums: If $m < n$ then
$$\sum_{k=m}^{n-1} a_k \, \Delta b_k = [a_k b_k]_m^n - \sum_{k=m}^{n-1} (\Delta a_k) b_{k+1}.$$

ii) Find the following indefinite sum:
$$\sum k^2.$$

d) Consider the (scalar) second-order difference equation
$$\Delta^2 x(k-1) = f(k, x(k)), \quad k = 1, \ldots, T, \qquad (1)$$
$$x(0) = 0, \qquad x(T+1) = 0. \qquad (2)$$

If there is a positive constant R such that
$$x f(k,x) > 0, \quad \text{for } k = 0, \ldots, T, \quad \text{and } \forall |x| \geq R,$$
then show all solutions must satisfy $|x(k)| < R$ for $k = 0, \ldots, T+1$.

2. a) Provide a paragraph or two outlining the motivation for studying "dynamic equations on time scales".

b) i) Define a time scale.

ii) Provide three examples of time scales and give an example of a set that is not a time scale.

c) Let $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^\kappa$. Show that if f is continuous at t and t is right-scattered then f is delta-differentiable at t with
$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

Please see over . . .

d) Let f and g be continuous, delta-differentiable functions. Prove the product formula for time scales with only **isolated** points:

$$(f(t)g(t))^\Delta = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)), \quad \forall t \in \mathbb{T}^\kappa.$$

e) Briefly explain why the product of two functions, fg, may not be twice delta-differentiable, even though f and g may be twice delta-differentiable in their own right.

f) For $f(t) = t^2$, find $f^\Delta(t)$ for:

i) $\mathbb{T} = \{\sqrt{n} : n \in \mathbb{N}_0\}$,

ii) $\mathbb{T} = \{n/2 : n \in \mathbb{N}_0\}$.

3. a) Prove that every right-dense (rd) continuous function has an anti-delta-derivative.

b) If $f \in C_{rd}$ and $t \in \mathbb{T}^\kappa$ then show that

$$\int_t^{\sigma(t)} f(s)\Delta s = \mu(t)f(t).$$

Hence show that if $[a,b]_\mathbb{T}$ consists of only isolated points and $a, b \in \mathbb{T}$ then

$$\int_a^b f(t)\Delta t = \sum_{t\in[a,b)_\mathbb{T}} \mu(t)f(t).$$

c) Evaluate the following delta integrals:

i)

$$\int_1^\infty \frac{1}{t^2}\Delta t, \quad \mathbb{T} = q^{\mathbb{N}_0} = \{q^0, q^1, q^2, \ldots\}, \quad q > 1,$$

ii)

$$\int_a^\infty \frac{1}{t\sigma(t)}\Delta t, \quad \mathbb{T} \text{ arbitrary}, \; 0 < a \in \mathbb{T}, \quad \sup \mathbb{T} = \infty,$$

iii)

$$\int_0^t s\ \Delta s, \quad t \in \mathbb{T} = [0,1] \cup [2,3].$$

iv)

$$\int_a^b (\mu(t) + t)e_\alpha(t,0)\Delta t, \quad \mathbb{T} \text{ arbitrary}, \quad a, b \in \mathbb{T}, \quad \alpha \in \mathcal{R}.$$

4. a) Consider the following dynamic IVP on arbitrary $\mathbb{T}$

$$y^\Delta = p(t)y, \quad p \in \mathcal{R}, \tag{3}$$
$$y(t_0) = 1, \quad t_0 \in \mathbb{T}. \tag{4}$$

Show that the only solution to (3), (4) is $e_p(t, t_0)$.

b) Consider the following dynamic IVP on arbitrary $\mathbb{T}$

$$\begin{aligned} y^{\Delta} &= p(t)y, \quad p \in \mathcal{R}, && (5) \\ y(t_0) &= y_0, \quad t_0 \in \mathbb{T},\ y_0 \in \mathbb{R}. && (6) \end{aligned}$$

Under which conditions will solutions to (5), (6): oscillate; be non-oscillatory; be strictly positive; be strictly negative?

c) i) Show that the solution to the dynamic (nonhomogeneous) IVP

$$\begin{aligned} y^{\Delta} &= -p(t)y^{\sigma} + f(t), \quad p \in \mathcal{R}, \quad f \in C_{rd}, \\ y(t_0) &= y_0, \quad t_0 \in \mathbb{T}, \quad y_0 \in \mathbb{R}, \end{aligned}$$

is

$$y(t) = \frac{y_0 + \int_{t_0}^{t} e_p(s,t_0) f(s) \Delta s}{e_p(t,t_0)}.$$

ii) Hence solve

$$\begin{aligned} y^{\Delta} &= -ty^{\sigma} + 1/e_p(t,t_0), \quad p \in \mathcal{R}, \\ y(t_0) &= 1, \quad t_0 \in \mathbb{T}. \end{aligned}$$

d) Let $N(t)$ be the number of plants of a certain species at time t. During the months of April until September, N grows exponentially according to $N' = N$. At the beginning of October, all plants suddenly die out, but the seeds remain in the ground and start growing again when they germinate in the following April with N now being doubled. An appropriate time scale to model N on is given by

$$\mathbb{T} = \cup_{k=0}^{\infty} [2k, 2k+1].$$

Explain why the model leads to the dynamic equation $N^{\Delta} = N$. If there is originally $N(0)$ number of plants then solve the dynamic IVP for N.

5. a) Let $r \in C_{rd}$ and $p \in \mathcal{R}^{+}$.

i) Show that if

$$r^{\Delta}(t) \le p(t) r(t), \quad \forall t \in [t_0, \infty)_{\mathbb{T}}, \quad t_0 \in \mathbb{T},$$

then

$$r(t) \le r(t_0) e_p(t,t_0), \quad \forall t \in [t_0, \infty)_{\mathbb{T}}.$$

ii) Consider the system of dynamic IVPs

$$\begin{aligned} x^{\Delta} &= f(t,x), \qquad t \in [t_0, \infty)_{\mathbb{T}}, && (7) \\ x(t_0) &= x_0, \qquad t_0 \in \mathbb{T}, \quad x_0 \in \mathbb{R}^n, && (8) \end{aligned}$$

where $f : [t_0, \infty)_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$. Prove the following theorem: If $\exists p \in C_{rd}([t_0, \infty)_{\mathbb{T}}; [0, \infty))$ such that

$$2\langle x, f(t,x)\rangle + \mu(t)\|f(t,x)\|^2 \leq p(t)\|x\|^2, \quad \forall (t,x) \in [t_0, \infty)_{\mathbb{T}} \times \mathbb{R}^n; \tag{9}$$

$$\lim_{t\to\infty} e_p(t, t_0) < \infty, \tag{10}$$

then all solutions x to (7), (8) are bounded on $[t_0, \infty)_{\mathbb{T}}$.

iii) Illustrate the applicability of the above result in (ii) by constructing a f and $\mathbb{T}$ such that the conditions of the theorem are satisfied.

b) Consider the (scalar) first-order dynamic BVP

$$x^\Delta = f(t,x), \quad t \in [a,c]_{\mathbb{T}}, \tag{11}$$

$$Mx(a) + Rx(\sigma(c)) = \alpha, \quad a, c \in \mathbb{T}, \quad \alpha \in \mathbb{R}^n, \tag{12}$$

where $f : [a,c]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is a continuous, nonlinear function; M, R and α are given constants in $\mathbb{R}$.

i) Reformulate the dynamic BVP (11), (12) in the fixed-point form

$$x(t) = P(x(t)), \quad t \in [a, \sigma(c)]_{\mathbb{T}},$$

where P is an approriate operator, making sure you state any conditions you are assuming in the process.

ii) Hence provide some general sufficient conditions (on f and possibly other components) such that all of the solutions x to the dynamic BVP (11), (12) are bounded *a priori* on $[a, \sigma(c)]_{\mathbb{T}}$.