MATH 1231 MATHEMATICS 1B 2007.

For use in Dr Chris Tisdell's lectures: Tues 11 + Thur 10 in KBT

Calculus Section 2: - ODEs.

1. Motivation

2. What you should already know

3. Types and orders of ODEs

4. What is a solution?

5. First–order ODEs

6. Second–order ODEs

7. MAPLE

Lecture notes created by Chris Tisdell & Peter Brown.

1. Motivation.

Why study differential equations?

How are they useful??

2. What you should already know.

At school you studied the solutions of (at least) two ODEs:

The *growth–decay* ODE

$$\frac{dP}{dt} = kP, \quad k = \text{const.}; \; P = P(t),$$

used in population change and radioactive decay,

Simple Harmonic Motion ODE

$$\frac{d^2x}{dt^2} = -n^2 x, \quad n = \text{const.}; \; x = x(t),$$

used to describe the motion of a weight on a spring or the approx. motion of a pendulum.

3. Types & orders of ODEs

The **order** of a differential equation is the highest derivative that appears in the equation. Since only *ordinary* derivatives, rather than partial derivatives, are involved, the equations are called *ODEs*.

Here are some types of ODEs:

4. What is a solution?

When given an ODE, the really big question that we will want to answer is:

However, first we need to understand what a solution to an ODE means.

A *solution* to an ODE is a **function** which is differentiable and which satisfies the given equation.

Ex. Consider

$$\frac{dy}{dx} = \frac{2}{x}y + x^3.$$

Show that $y(x) = \frac{x^4}{2} - 5x^2$ is a solution.

Initial value problems

Ex. Solve $\dfrac{d^2y}{dx^2} = e^x$ given $y(0) = 3$, $y'(0) = 2$.

In the above example, we were give an ODE and some extra piece of information, known as "initial conditions".

In the simple harmonic motion problem, you are often given the initial displacement and the initial velocity. That is, you are given $x(0)$ and $x'(0)$.

When thinking of solving a given initial value problem, the following questions arise:

a) Does the initial value problem actually have a solution?

b) Does it have a unique solution?

c) if initial values are given at $x = a$ how far either side of a does the solution extend?

Ex. Given $\frac{dy}{dx} = \frac{1}{x}$ with $y(1) = 2$. The solution does not exist at $x = 0$.

Ex. Solve $y' = \sqrt{y}$, $y(0) = 0$. There are two different solutions $y(x) = 0$ and $y(x) = \frac{x^2}{4}$.

4. First order equations.

There is no universal method for solving ODEs of first (or any other) order.

We will develop and apply a collection of techniques for solving certain types of equations.

In this course we will be studying how to solve four basic classes of first order equations:

- separable equations
- homogeneous equations
- linear equations
- exact equations.

Separable Equations.

Are of the type:

The solution method is:

Ex. Solve the initial value problem

$$\frac{dy}{dx} = y^2(1 + x^2)$$

with initial value $y(0) = 1$.

Ex: Solve

$$\frac{dy}{dx} = x\sqrt{1 - y^2}.$$

(Ans: $y = \sin(\frac{x^2}{2} + C)$.)

Homogeneous Equations.

Are of the type

And have solution method:

Given an ODE, can I manipulate it so that it is of the form

Ex. Is the ODE $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ homogeneous?

Ex. What about $\frac{dy}{dx} = \frac{y^2 + 2xy + x^2}{x^2 - 3y^2 + 7x}$?

More formally, we can write the given ODE as

$$\frac{dy}{dx} = F(x, y).$$

This is homogeneous if for all real numbers λ, we have

$$F(\lambda x, \lambda y) = F(x, y).$$

Ex. Solve

$$\frac{dy}{dx} = \frac{2x + y}{x}.$$

Ex. Solve, for $x > 0$,

$$x^2 + y^2 - 2xy\frac{dy}{dx} = 0.$$

Linear Equations.

A linear first order equation has the form:

$$\frac{dy}{dx} + f(x)y = g(x)$$

where f and g are given functions.

Essentially they are characterised by the lack of any non-linear terms in y or y',e.g. there are no y^2 terms or e^y terms or $\sqrt{y}$ terms, and no terms involving y times its derivative.

The solution method for linear equations is as follows.

Ex. Solve

$$\frac{dy}{dx} + 3y = e^{-x}.$$

Ex. Solve, for $x > 0$,

$$\frac{dy}{dx} + \frac{y}{x} = x$$

subject to $y = 4$ when $x = 3$.

Ex. Solve, for $x > 0$,

$$x\frac{dy}{dx} + (x+1)y = 2.$$

Applications of ODEs

Suppose we wish to measure the temperature of a very hot object, so hot in fact that our thermometer (which only works from -40^o to 100^oC) can't be used. How can we find out the temperature of the object?

We have to assume Newton's law of cooling, which says that:

the rate of decrease of temperature is proportional to the difference between the temperature of the object and its surroundings i.e.

$$\frac{dT}{dt} = k(T - A) \quad \begin{cases} A = \text{ambient temp.} \\ T = \text{temp. after time } t \text{ in mins} \\ k = \text{unknown const.} \end{cases}$$

Further suppose we have the data,

$$A = 20^o,\ T(6) = 80^o,\ T(8) = 50^o.$$

Observe that although the equation is linear, it is also separable and this is the easiest way to solve it. Separating the variables we have, Ex. An investor has a salary of \$60,000 per year and expects to get an annual increase of \$1000 per year. Suppose an initial deposit of \$1000 is invested in a program that pays 8% per annum and additional deposits are added yearly at a rate of 5% of the salary. Find the amount invested after t years.

We can approximate by assuming that the interest is calculated continuously and that deposits are made continuously.

Let $x(t) =$ amount of money invested at time t (in years), then

$$\frac{dx}{dt} = 0.08x + 0.05(60000 + 1000t)$$

rate of increase of investment
$=$ 8% of investment $+$ 5% of salary

Solving we have

$$x(t) = -625t - \frac{90625}{2} + Ce^{0.08t}.$$

Using $t = 0, x = 1000$ we obtain

$$x(t) = \frac{92625}{2}e^{0.08t} - (625t + \frac{90625}{2}).$$

We can now predict, with accuracy, the amount invested at any future time.

Eg, when $t = 3$ we have $x = 11687.22$,

at $t = 10$ we have $x = 51507.86$

and when $t = 30$ we have $x = 446448$.

How would you solve the problem

$$y \cdot (y')^3 = a, \quad y(\xi) = \nu$$

where a, ξ and ν are known constants? This IVP arises in the analysis of rotationally symmetric bodies of smallest wave drag at hypersonic flow.

The solution is

$$y(x) = \left[(4/3) \cdot a^{1/3} \cdot (x - \xi) + \nu^{4/3}\right]^{3/4}.$$

See J. T. Heynatz, Application of the Newton method to the so–called integral equation method in transonic flow, Acta Mechanic 138 (1999), 123–127.

Modelling of phenomena.

Many real–life problems can be analysed and solved by attempting to convert them into mathematics and analysing the resultant equations.

In doing so, a number of assumptions have to be made and a theoretical framework set up which attempts to reflect what is happening in the real world. Such a framework is called a **mathematical model**. The reliability of that model depends on how well it predicts what actually happens in the real world.

Because derivatives are used to measure rates of change, it is not surprising that differential equations often turn up in modelling problems.

In the quest for simplicity and solvability, our equations may lose some important aspects of the real–world model.

In developing a mathematical model we try to:

- accurately describe the data we have

- decide exactly what information we wish to extract from our model

- decide which variables in the model are dependent and which are independent

- describe how the dependent variables change as the independent variables change (which may lead to a differential equation.)

Some Population Models

Model 1: A population has initial size of 2 million and a growth rate of 2% per annum. Find a model for the population size.

Let P be the population size. We can approximate the solution by the d.e.

Criticisms of the Model:

1) In this model we assume that the population can continue to grow indefinitely.

2) External factors such as disease, natural disaster etc have been ignored.

Model 2: We try to overcome criticism (1) by assuming that there is a maximum population P_m which when exceeded causes a population decrease and when not exceeded causes a population increase.

Try the d.e.

Criticisms

1) Observe that as $P \to 0$ we have

$$\frac{dP}{dt} \to kP_m$$

which means that when the population is very small, it still has a (possibly large) positive growth rate !

2) As before external factors are ignored.

Model 3: To overcome criticism (1), we try

Mixing Problems:

Ex. A martini drink is, in essence, a mixture of the two liquids *gin* and *vermouth*. James Bond insists his martinis are prepared as follows. Initially, 40 cc of gin are placed in a large container. Then gin is poured into the container at the rate of 2 cc/sec, and at the same time vermouth at the rate of 6 cc/sec. The mixture is constantly shaken (not stirred) and flows out at a rate of 4 cc/sec.

i) Find an expression for the volume of vermouth in the container at time t seconds after the pouring commences.

ii) James likes his martini to have 2 parts of gin to 3 of vermouth how many seconds should elapse before he stops pouring and inserts a cocktail glass in the outflow from the container?

i) The volume of liquid in the container at time t is $40 + 4t$ cc. Let V be the volume of Vermouth in the tank at time t, so the *proportion* by volume of vermouth in the tank at time t is $\dfrac{V(t)}{40 + 4t}$. Then ii)

Exact equations.

Many first order ODEs can be written in the form $\dfrac{dy}{dx} = -\dfrac{M(x,y)}{N(x,y)}$ or as

$$M(x,y)\,dx + N(x,y)\,dy = 0 \qquad (*).$$

This is reminiscent of the total differential of a function of two variables you saw in MATH1131.

Suppose we have a function $F(x,y) = C$ where C is a constant. Taking the total differential we have

$$\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = 0$$

Now **if**

$$M = \frac{\partial F}{\partial x} \text{ and } N = \frac{\partial F}{\partial y}, \qquad (**)$$

then reversing the above steps, we could conclude that the solution to the differential equation $(*)$ was

$$F(x,y) = C.$$

Differentiating the first expression in $(**)$ with respect to y and second with respect to x and recalling that for 'nice' functions the order in which we partially differentiate doesn't matter, we have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

and this will be the condition for exactness. That is, given an ODE of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

we say that it is exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

If it is exact, then from the above, it must have a solution of the form $F(x, y) = C$ where $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$. So we can use these equations to reconstruct $F(x, y)$ and so obtain our solution.

Ex. Solve

$$(2x + y + 1)dx + (2y + x + 1)dy = 0.$$

Ex. Solve $\dfrac{dy}{dx} = -\dfrac{2xy + 3x^2y^2}{x^2 + 2x^3y + 1}$ which could also be written as

$$(2xy + 3x^2y^2)\,dx + (x^2 + 2x^3y + 1)\,dy = 0.$$

6. Second order ODEs

The second order linear ODE with constant co–efficients, has the form

$$y'' + ay' + by = f(x)$$

where y is a function of x which we have to determine. If $f(x) = 0$, then we say that the equation

$$y'' + ay' + by = 0 \qquad (*)$$

is *homogeneous*.

Note carefully that we are using this word in a **different** sense to that which we used it for in regard to first order equations. We will deal firstly with the homogeneous equation.

Firstly observe that if we have two solutions $y_1(x)$ and $y_2(x)$ to $(*)$, then any linear combination of these is also a solution, since

Now in order to solve the homogeneous equations $y'' + ay' + by = 0$, we seek a function y which does not change a great deal when differentiated. We try therefore $y = Ae^{\lambda x}$, where A is a non-zero constant.

Ex. Solve $y'' - 5y' + 6y = 0$. Try $y = Ae^{\lambda x}$.
Substitution yields

The quadratic is called the

characteristic equation

associated with the ODE. We normally go straight to the characteristic equation, solve it and simply write down the general solution.

In general the characteristic equation for

$$ay'' + by' + cy = 0 \quad \text{is} \quad a\lambda^2 + b\lambda + c = 0.$$

Ex: Solve the initial value problem

$$y'' - 4y' - 5y = 0, \text{ with } y(0) = 2, \ y'(0) = 10.$$

Repeated Roots:

Ex. Solve

$$y'' - 4y' + 4y = 0.$$

The char. eqn. has repeated root, $\lambda = 2$, so a solution is

$$y_1 = Ae^{2x}.$$

We now try the following trick. We look for a solution of the form

$$y_2 = v(x)y_1 = v(x)e^{2x}.$$

Substitution leads us to $e^{2x}v''(x) = 0$ and so $v(x) = Bx$, thus $y_2(x) = Bxe^{2x}$.

The sort of technique we used above works in general and so if the characteristic equation has a single repeated root, λ, then the general solution is

$$y = Ae^{\lambda x} + Bxe^{\lambda x}.$$

Ex. Solve

$$y'' + 6y' + 9y = 0$$

subject to $y(0) = 0$, $y'(0) = 5$.

Complex Roots:

Finally, the third possibility is that the roots are complex, and we know that complex roots of a real quadratic occur in pairs as

$$\lambda = \alpha \pm i\beta.$$

Thus, as above we have solution

$$\begin{aligned} y &= Ce^{(\alpha+i\beta)x} + De^{(\alpha-i\beta)x} \\ &= (C+D)e^{\alpha x}\cos\beta x + i(C-D)e^{\alpha x}\sin\beta x. \end{aligned}$$

(By Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.)

Now the constants C and D are arbitrary, so to get real solutions, we choose C, D such that $A = C + D$ and $B = i(C - D)$ are both real and so the general real solution is

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x).$$

Ex. Solve

$$y'' - 2y' + 5y = 0.$$

The characteristic equation has solution

To summarise, we solve the characteristic equation of

$$y'' + ay' + by = 0$$

which is

$$\lambda^2 + a\lambda + b = 0.$$

If

• roots are real and distinct, λ_1, λ_2, the general solution is

• roots are equal, λ, the general solution is

• roots are complex, $\lambda = \alpha \pm i\beta$, the general solution is

The Non-Homogeneous Case:

Ex. Solve

$$y'' - 5y' + 6y = 2x + 3.$$

Ex: Solve

$$y'' - 5y' + 6y = 12e^{5x}.$$

We can construct a table of what to try as a particular solution for given $f(x)$.

$f(x)$
$P(x)$, a nth deg. polyn.
$P(x)e^{ax}$
$P(x)\cos ax$
$P(x)\sin ax$
$P(x)e^{ax}\sin bx$ or $P(x)e^{ax}\cos bx$

Respective Guess for y_p.
$Q(x)$, nth deg. polyn.
$Q(x)e^{ax}$
$Q_1(x)\cos ax + Q_2(x)\sin ax$
$Q_1(x)\cos ax + Q_2(x)\sin ax$
$Q_1(x)e^{ax}\cos bx + Q_2(x)e^{ax}\sin bx$

***Care must be taken however, when using the above table as the following examples will show. ***

Ex. Solve $y'' - 5y' + 6y = 12e^{2x}$.

An even more unpleasant example is:

Ex. Solve $y'' - 4y' + 4y = 2e^{2x}$.

Thus, as a general rule, if the right hand side of the equation has a function which is already in the kernel (i.e. one of the homogeneous solutions), we multiply by x until the resulting function is no longer a solution to the homogeneous equation.

Ex. Solve $y'' + y = \cos x$, such that $y(0) = 3, y'(0) = 0$

Applications:

Ex. An object falls from rest under gravity with wind resistance proportional to its speed. It's terminal velocity is $100m/s$. How long does it take to fall 1000m?

The governing equation is

When $x = 1000$, we need to solve for t, which can be done using MAPLE and we get ≈ 18.546 seconds.

Forced Vibrations:

All structures have natural frequencies of vibration. If an external agent causes them to vibrate at or near one of these frequencies, large oscillations build up and *resonance* occurs. This can cause such disasters as the collapse of bridges and other structures.

To understand the mathematics of these forced vibrations, let us look at a fairly simple example of an object of mass m suspended from a spring length ℓ_0 mounted from a point A.

When the object is placed on the spring, it extends a distance e to the equilibrium position. At this point, we have $mg = ke$, by Hooke's law, where k is a constant (called the *stiffness* of the spring).

Let x be the displacement of the object from the equilibrium point, then using Newton's second law (and the fact that the tension T in the spring is proportional to the displacement from the equilibrium point), we have

$$m\frac{d^2x}{dt^2} = mg - T = mg - k(e + x).$$

Now using $mg = ke$ and writing $\frac{k}{m} = \omega^2$ we obtain the homogeneous differential equation

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

which has solution,

The top end A of the spring is now forced to vibrate so that its displacement at time t is given by $y = a \sin \Omega t$. In other words we introduce a new vibrating force into the system. The extension T now becomes $T = k(e+x-y)$ and so the differential equation becomes

$$m\frac{d^2x}{dt^2} = mg - T = mg - k(e + x - y).$$

Again we use $mg = ke$ and replace $\frac{k}{m}$ by ω^2 giving

$$\frac{d^2x}{dt^2} + \omega^2 x = \omega^2 a \sin \Omega t.$$

This is now a non-homogeneous equation. IF $\omega \neq \Omega$ then the homogeneous equation has solution, $x_h = A \cos \omega t + B \sin \omega t$ and we can try for a particular solution of the form $x_p = C \cos \Omega t + D \sin \Omega t$. Substituting and solving we have the general solution

This is simply another oscillating motion.

However, if $\omega = \Omega$, that is, when the driving vibration has the same frequency as the natural frequency of the system, then the form of the particular solution needs to be changed to

Again, substituting and solving for C and D, we obtain the solution

Hence as t increases the displacement increases without bound and the system becomes unstable.

7. MAPLE

The following command is used in MAPLE to solve ODE's (if possible).

```
>dsolve(deqn, y(x));
```

For example

```
>dsolve(diff(y(x),x\$2)-y(x) = 1, y(x));
```

$$y(x) = -1 + \_C1\,\mathrm{exp}(x) + \_C2\,\mathrm{exp}(-x)$$

```
>dsolve({diff(v(t),t) + 2*t = 0, v(1) = 5},
v(t));
```

$$v(t) = -t^2 + 6$$