

MATH 1231 MATHEMATICS 1B 2007.

For use in Dr Chris Tisdell's lectures: Tues 11 + Thur 10 in KBT

Calculus Section 1: - Integration.

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Lecture notes created by Chris Tisdell & Peter Brown.

1. Motivation.

Why study integration?

How is it useful??

2. What you should already know.

You studied integration at school and in MATH 1131. You should be comfortable with:

- using a table of integrals
- integrating by inspection
- integration by parts
- integration by substitution.

Just in case you've forgotten these techniques, I have included some examples to refresh your memory and to test your skills.

You will find this course *much easier* if you can integrate by parts and integrate by inspection with ease. Your hard work will pay-off later in the session!!

2 (i). Integration by Parts.

Ex. For $x > 0$ find $\int x^3 \ln x \, dx$.

Ex. $I = \int e^x \cos x \, dx =$

Here are some problems to refresh your skills. Integrate the following by parts.

$$\begin{array}{ll} \text{a. } \int x^2 e^{-x} dx & \text{b. } \int x^3 \ln x dx \\ \text{c. } \int \frac{x}{\cos^2 x} dx & \text{d. } \int \frac{(\ln x)^2}{x^2} dx \\ \text{e. } \int e^x \cos x dx & \text{f. } \int \ln x dx \\ \text{g. } \int \tan^{-1} x dx \end{array}$$

Answers:

$$\begin{array}{ll} \text{a) } -e^{-x}(x^2 + 2x + 2) & \text{b) } \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 \\ \text{c) } x \tan x + \ln(\cos x) & \text{d) } -\frac{(\ln x)^2}{x} - 2\frac{\ln x}{x} - \frac{2}{x} \\ \text{e) } \frac{1}{2}e^x(\sin x + \cos x) & \text{f) } x \ln x - x \\ \text{g) } x \tan^{-1} x - \frac{1}{2} \ln(x^2 + 1) \end{array}$$

(N.B. The $+C$ has been omitted).

2 (ii). Integration by inspection.

When confronted by an integral, you should always see if the integral can be 'guessed' and then fine tuned.

Ex. Find $\int x(x^2 + 1)^{10} dx$.

Ex. Find $\int \frac{x}{\sqrt{x^2 - 4}} dx$.

Ex. $\int \sin t \cos^5 t dt = -\frac{\cos^6 t}{6} + C$.

Observe that this method **ONLY** works for integrals of the form

$$\int f(g(x))g'(x) dx$$

when the derivative $g'(x)$ is present in the integrand up to a constant.

You can “correct” your guess by multiplying or dividing by a constant.

$$\int x^4(x^3 + 1)^{10} dx$$

cannot be ‘guessed’ and fine tuned since the derivative of $x^3 + 1$ is not in the integrand.

Here are some problems to refresh your skills. Evaluate each of the following integrals by inspection.

a. $\int x e^{2x^2} dx$	b. $\int x \sin(x^2) dx$
c. $\int x^2 \cos(2x^3) dx$	d. $\int \frac{x}{5x^2 - 11} dx$
e. $\int \sin x \cos^3 x dx$	f. $\int \frac{dx}{x \ln x}$
g. $\int \frac{x + 2}{\sqrt{x^2 + 4x + 7}} dx$	h. $\int x \sqrt{1 + x^2} dx$
i. $\int x^2 \sqrt{9 - 4x^3} dx$	j. $\int \frac{x^2}{\sqrt{9 - 4x^3}} dx$
k. $\int \frac{x^3}{(1 + x^4)^3} dx$	l. $\int \frac{\sec^2 x}{\tan^4 x} dx$
m. $\int \frac{\cos x}{\sin^3 x} dx$	n. $\int e^{2x} (4 + 3e^{2x})^{1/3} dx$
o. $\int \frac{1}{x (\ln x)^5} dx$	

If you get stuck then use substitution as a last resort.

a) $\frac{1}{4} e^{2x^2}$	b) $-\frac{1}{2} \cos(x^2)$	c) $\frac{1}{6} \sin(2x^3)$
d) $\frac{1}{10} \ln 5x^2 - 11 $	e) $-\frac{1}{4} \cos^4 x$	f) $\ln(\ln x)$
g) $\sqrt{x^2 + 4x + 7}$	h) $\frac{1}{3} (1 + x^2)^{\frac{3}{2}}$	i) $-\frac{1}{18} (9 - 4x^3)^{\frac{3}{2}}$
j) $-\frac{1}{6} (9 - 4x^3)^{\frac{1}{2}}$	k) $-\frac{1}{8(1 + x^4)^2}$	l) $\frac{-1}{3 \tan^3 x}$
m) $\frac{-1}{2 \sin^2 x}$	n) $\frac{1}{8} (4 + 3e^{2x})^{\frac{4}{3}}$	o) $\frac{-1}{4(\ln x)^4}$

(N.B. The $+C$ has been omitted).

3. Useful integrals.

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}$$

$$\int \frac{1}{\sqrt{a^2 + x^2}} dx = \sinh^{-1} \frac{x}{a}$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{a} \tanh^{-1} \frac{x}{a}$$

These four should be learnt carefully.

4. Integrals of trig functions.

You should already be familiar with the technique to find the primitives of $\sin^2 x$ and $\cos^2 x$.

$$\int \cos^2 x \, dx =$$

$$\int \sin^2 x \, dx =$$

Integrals of the form $\int \cos^m x \sin^n x dx$

If m or n are odd then we can take a $\sin x$ or $\cos x$ from the odd power term use the identity $\cos^2 x + \sin^2 x = 1$ to transform the integral into one which can be guessed.

Ex. $\int \cos^4 x \sin^3 x dx =$

Ex. $\int \cos^5 x \sin^5 x dx =$

If **both** m and n are even, then the integral is slightly harder. We will see shortly a recursive method for dealing with such an integral, but for smaller values of m and n one can use the identities $\cos^2 x = \frac{1+\cos 2x}{2}$ and $\sin^2 x = \frac{1-\cos 2x}{2}$ again.

Ex. Evaluate $\int \cos^4 x \sin^2 x dx$

Integrals of the form $\int \cos mx \sin nx \, dx$

These type are best done using the so-called *product to sum* formulae

$$\sin A \cos B =$$

$$\sin A \sin B =$$

$$\cos A \cos B =$$

$$\text{Ex. } \int \sin 3x \sin 2x \, dx =$$

On integrals of type $\int \tan^m x \sec^n x dx$

If n is even, then $\sec^n x = \sec^2 x \sec^{n-2} x$ and we can substitute $\sec^2 x = 1 + \tan^2 x$ in the latter term, leading to guessable integrals.

Ex. $\int \tan^2 x \sec^4 x dx =$

If m is odd then take out one of the $\tan x$ terms and replace the rest using $\tan^2 x = \sec^2 x - 1$ and again we get integrals which can be guessed, recalling that the derivative of $\sec x$ is $\sec x \tan x$.

Ex. $\int \tan^3 x \sec^3 x dx =$

If m is even then we can use the trig identity to reduce the integrand to a power of $\sec x$ and use the reduction formula in the next section.

Note carefully the following integral for $\sec x$.

5. Reduction formulae

A reduction formula is a recursive formula for a family of integrals. In other words, an integral involving a power is expressed in terms of similar integrals involving a smaller power.

For example, we shall shortly show that, for $n \geq 2$,

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx.$$

We can then apply the same formula to the second integral and reduce it down to an integral involving $\sin^{n-4} x$ and so on until the power becomes small enough to manage.

We can use a subscript notation to write the above as,

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx, \quad I_n = \frac{n-1}{n} I_{n-2}.$$

A formula such as the one above is called a *reduction formula*.

Ex. Use the reduction formula above to find $\int_0^{\pi/2} \sin^7 x \, dx$.

Ex. Find the reduction formula for $I_n = \int x^n e^x \, dx$.

Ex. Find the reduction formula for $I_n = \int_0^{\frac{\pi}{2}} \sin^n x$.

The reduction formula for the integral of $\cos^n x$ is similar.

$$\text{Ex. } I_n = \int_0^{\frac{\pi}{4}} \sec^n x.$$

A similar method is used to get the reduction formula for the integral $\int_0^{\frac{\pi}{4}} \tan^n x$.

Ex. [Q1, Class Test 1, 2002] Let

$$I_n = \int_0^{\pi/4} \tan^n \theta \sec \theta \, d\theta.$$

Show that

$$I_n = \frac{1}{n} \left(\sqrt{2} - (n-1)I_{n-2} \right), \quad \text{for } n \geq 2.$$

Note that

$$\frac{d}{d\theta} \sec \theta = \sec \theta \tan \theta.$$

A Two parameter recurrence:

$$I_{m,n} = \int_0^{\frac{\pi}{2}} \cos^m x \sin^n x dx$$

In a similar way, one could also obtain the recurrence

$$I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}$$

for $n \geq 2$. In applying the above formulae we must reach one of $I_{1,1} = \frac{1}{2}$, $I_{1,0} = I_{0,1} = 1$ or $I_{0,0} = \frac{\pi}{2}$.

You are not expected to memorise this formula.

6. Trig & Hyperbolic Substitutions.

Integration by substitution was covered in MATH 1131. It involves making a 'sensible' change of variable which reduces the integral down to either a standard or a 'guessable' one.

Ex. Find $\int \log(\cos x) \tan x \, dx$, put $u = \cos x$.

Integrals involving square roots of quadratics can be worked out using the following trigonometric or hyperbolic substitutions.

$\sqrt{a^2 - x^2}$	try $x = a \sin \theta$
$\sqrt{a^2 + x^2}$	try $x = a \tan \theta$ or $x = a \sinh \theta$
$\sqrt{x^2 - a^2}$	try $x = a \sec \theta$ or $x = a \cosh \theta$

$$\text{Ex. } I = \int \sqrt{1 - x^2} dx.$$

$$\text{Ex. } I = \int_0^4 x^2 \sqrt{16 - x^2} dx.$$

$$\text{Ex. } I = \int \frac{dx}{(a^2 + x^2)^{3/2}}.$$

$$\text{Ex. } I = \int \frac{dx}{x^2 \sqrt{(x^2 - 1)}}. \text{ Put } x = \sec \theta$$

(See that this last integral can also be done using a $\cosh \theta$ substitution but it is not easy.)

7. Partial Fractions.

A *rational function* is a function of the form $\frac{p(x)}{q(x)}$ where p and q are polynomials, for exam-

ple $\frac{1}{x}$, $\frac{x^3 + 2}{x^6 - 1}$ are rational functions.

We ensure that the degree of the numerator is less than the degree of the denominator. If not then we may need to divide.

$$\text{Ex. } \frac{x^3 - x + 1}{x^2 + 3x - 1} = x - 3 + \frac{9x - 2}{x^2 + 3x - 1}.$$

$$\text{Ex. } \frac{4x - 5}{2x + 3} = 2 - \frac{11}{2x + 3}.$$

Supposing our rational function is thus reduced, we now factorise the denominator over the real numbers, generally into linear/quadratic factors.

Factors are Linear:

$$\text{Ex. } f(x) = \frac{2x - 1}{x^2 + 5x + 6} = \frac{2x - 1}{(x + 3)(x + 2)}$$

In practise you use any method you can to find the co-efficients. Usually, substitute some values of x and then equate co-efficients.

$$\text{Ex. } f(x) = \frac{2x^2 + 2x + 6}{(x - 1)(x + 1)(x - 2)} =.$$

Using the partial fraction form we can now perform the integral

$$\int \frac{2x^2 + 2x + 6}{(x - 1)(x + 1)(x - 2)} dx =$$

Repeated Linear Factors:

e.g. $\frac{x + 3}{(x + 2)^3}$.

We could then evaluate $\int \frac{x + 3}{(x + 2)^3} dx$ to get

$$\frac{-1}{(x + 2)} - \frac{1}{2(x + 2)^2} + C.$$

In general, each factor in the denominator of the form $(x - a)^k$ gives rise to an expression of the form

$$\frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \dots + \frac{A_k}{(x - a)^k}.$$

Ex. Evaluate $\int \frac{dx}{x(x + 1)^3}$.

Thus $\int \frac{dx}{x(x + 1)^3} =$

Ex. [Q2, Class Test 1, 2003] Find

$$\int \frac{8x + 9}{(x - 2)(x + 3)^2} dx.$$

Irreducible Quadratic Factors:

Suppose the denominator of our rational function is a quadratic which does not factor over the real numbers. For example,

$$f(x) = \frac{2x + 1}{x^2 + 4x + 5}.$$

To find the integral of such a function, we *make* the numerator equal to the derivative of the denominator, leading to a logarithm and complete the square on the denominator of the remaining term leading to an inverse tangent function. *viz*

$$\int \frac{2x + 1}{x^2 + 4x + 5} dx =$$

(N.B. If the quadratic does have real but irrational roots, one can use this same method to avoid nasty partial fractions. The second factor will then lead to an inverse tanh.

e.g.

$$\begin{aligned} & \int \frac{2x + 1}{x^2 + 4x - 6} dx \\ &= \int \frac{2x + 4}{x^2 + 4x - 6} + \frac{3}{10 - (x + 2)^2} dx \\ &= \ln|x^2 + 4x - 6| + \frac{3}{\sqrt{10}} \tanh^{-1} \left(\frac{x + 2}{\sqrt{10}} \right) + C. \end{aligned}$$

Ex. [Q1, Class Test 1, 2002] Find

$$\int \frac{x}{x^2 + 2x + 10} dx.$$

You are given that

$$\int \frac{du}{u^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C.$$

Combinations of the Rules:

One can combine these rules to cover a variety of situations.

Ex. Find $\int \frac{x + 6}{x(x^2 + 2x + 3)} dx$.

In general, an irreducible quadratic factor

$$x^2 + ax + b$$

in the denominator will give rise to a term of the form

$$\frac{Ax + B}{x^2 + ax + b}$$

and more generally again, a factor

$$(x^2 + ax + b)^k$$

in the denominator will give rise to terms of the form

$$\frac{A_1x + B_1}{x^2 + ax + b} + \frac{A_2x + B_2}{(x^2 + ax + b)^2} + \dots + \frac{A_kx + B_k}{(x^2 + ax + b)^k}.$$

Ex. Find $\int \frac{x}{(x+1)^2(x^2+1)} dx =$

8. Rationalising substitutions

The above methods using partial fractions allow us to integrate (in principle) any rational function, so we often make a change of variable in an integral that will lead us to some rational function.

Ex. Find $\int \frac{dx}{1 + x^{1/4}}$

Ex. Find $\int \frac{dx}{\sqrt{e^{2x} - 1}}$.

9. Weierstrass substitutions

The *Weierstrass* substitution is very useful for integrals involving a single term in $\sin x$ and/or $\cos x$ in the denominator.

The *Weierstrass* substitution is $t = \tan \frac{x}{2}$.

Recall, that $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, and that $dx = \frac{2 dt}{1+t^2}$.

You should know how to derive these formulae.

Who was Weierstrass??

Ex. Find $\int \frac{dx}{1 + \cos x + \sin x}$.

Ex. Show that $\int_0^{\frac{\pi}{2}} \frac{dx}{2 + \cos x} = \frac{\pi}{3\sqrt{3}}$.

10. MAPLE

The following examples show how to use MAPLE to perform partial fraction decompositions.

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> convert(x^2/(x+2), parfrac, x);
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$$x - 2 + \frac{4}{x + 2}$$

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> convert(x/(x-b)^2, parfrac, x);
```

$$\frac{b}{(x - b)^2} + \frac{1}{x - b}$$