The objective of this section is to become familiar with the theory and application of slicing techniques and their application to

- volume;
- area;
- surface area;
- and arc-length.

By the end of this section students will be able to solve a range of problems involving the above ideas.
1. Volumes by Slicing:

Recall, that the volume of any prism (i.e. a solid with uniform cross section) is given by the area of the cross-section \( \times \) the height. We can exploit this idea for solids which do not have uniform cross-section.
Suppose we put an axis through the solid (in whatever place is most convenient) and slice the solid perpendicular to this axis at a point $x$ where $a \leq x \leq b$. Let $V(x)$ be the volume of the solid from $a$ to $x$, and $A(x)$ be the cross-sectional area at the point $x$. Thus the volume of a slice from $x$ to $x + \Delta x$ will be $V(x + \Delta x) - V(x)$ which will be approximately $A(x)\Delta x$. Thus

$$A(x) \approx \frac{V(x + \Delta x) - V(x)}{\Delta x}.$$ 

Now if the limit as $\Delta x \to 0$ exists, we have

$$A(x) = \frac{dV}{dx}$$

and so, if $A(x)$ is integrable, we have

$$Volume \ of \ the \ solid \ = \int_a^b A(x) \, dx.$$
From this we can obtain the standard formula for volume of revolution.

**Theorem:** If \( f^2 \) is integrable, the volume of the solid obtained by rotating the curve 

\[ y = f(x) \]

(for \( f(x) \geq 0 \)) about the \( x \)-axis between \( x = a \) and \( x = b \) is given by 

\[ \int_a^b \pi(f(x))^2 \, dx. \]

Proof: The cross section of a slice at distance \( x \) from the origin is given by \( \pi(f(x))^2 \) and result follows.
Ex. Find the volume of the solid obtained by rotating the curve $y = x^2$ about the $x$-axis between the points $x = 0$ and $x = 2$. 
**Similar Areas:** We can use the idea of similar shapes to find volumes.

Ex. Find the volume of a right circular cone with height $h$ and radius $r$. 

The practical steps involved are:

1. Draw a clear sketch of the solid and a typical cross-section.

2. Find $A(x)$.

3. Find $a$ and $b$.

4. Integrate.
Ex. Find the volume of the solid generated when the area bounded by the \(x\)-axis and the curve \(y = -x^2 + 4x - 3\) is rotated about the \(y\)-axis.
Ex. Find the volume of the torus obtained by rotating $(x - a)^2 + y^2 = b^2$ (where $0 < b < a$) around the $y$-axis.
2. **Arc Length:**
We seek to find a formula for arc length of a given curve, \( y = f(x) \).

Take a small piece of arc length \( \Delta s \) between the points \( x \) and \( x + \Delta x \) on the curve and draw an approximate triangle there. By Pythagoras’ theorem we have

\[
(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2. \tag{*}
\]

Now dividing by \( (\Delta x)^2 \) and assuming the necessary limits exist, we have, using the same basic idea we used for slicing,

\[
s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx.
\]

provided \( f \) is differentiable and the integral exists.
Ex. Find the arc length of the curve

\[ y = \frac{1}{4}x^2 - \frac{1}{2}\log x \]

from \( x = 1 \) to \( x = 2 \).
Parametric Formula for Arc Length.

Ex. Find the arc length of the cycloid

\[ x(t) = a(t - \sin t), \quad y(t) = a(1 - \cos t) \]

for \( 0 \leq t \leq 2\pi \).
Ex. Find the arc length of the curve

\[ x = \int_0^t \frac{\sin s}{\sqrt{s}} \, ds, \quad y = \int_0^t \frac{\cos s}{\sqrt{s}} \, ds, \]

for \( 0 \leq t < 10 \).
Surface Area.
The problem of finding the surface area of a solid is not easy. The general question will be left 'till second year. We will instead concentrate on the problem of finding the surface area of a solid of revolution (about the $x$-axis).

The formula we will obtain is essentially based on the following simple result from school, that the area of the frustum of a cone is $\pi(R + r)s$ where $R$ and $r$ are the radii and $s$ is a the slant height.
Suppose that \( y = f(x) \) is rotated around the \( x \)-axis between \( x = a \) and \( x = b \). We slice the surface into strips and approximate each strip by a frustum of a cone having radii \( y \) and \( y + \Delta y \), with slant height being a ‘small’ piece of arc length which we can approximate as \( \sqrt{(\Delta x)^2 + (\Delta y)^2} \).

Thus the area is

\[
\pi(2y + \Delta y)\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x.
\]

Summing the areas of the strip and letting \( \Delta x, \Delta y \to 0 \), we obtain

\[
\text{Surface Area } = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx.
\]
Ex. Find the the surface area of the cone radius $r$ height $h$. This is done by rotating the
line $y = \frac{rx}{h}$, where $0 \leq x \leq h$, about the $x$- axis. We get $\pi r \sqrt{r^2 + h^2} = \pi rl$, where $l$ is the slant
height of the cone.
Ex. Find the surface area of the solid obtained by rotating \( y = \sqrt{2-x} \) between \( x = 0 \) and \( x = 2 \) about the \( x \)-axis.
Ex. Find the surface area of the solid obtained by rotating $y = \frac{x^2}{2}$ about the $x$-axis between $x = 0$ and $x = 1$. 
Average Value of a Function;
Suppose that $f$ is an integrable function defined on $[a, b]$. Define

$$ F(x) = \int_a^x f(t) \, dt. $$

From MATH1131, we know that $F$ is differentiable and so we can apply the Mean Value Theorem to $F$ on $[a, b]$ to obtain

$$ \frac{F(b) - F(a)}{b - a} = F'(c), \quad \text{for some } c \in (a, b). $$

Thus

$$ \frac{1}{b - a} \int_a^b f(t) \, dt = f(c) $$
or

$$ \int_a^b f(t) \, dt = (b - a) f(c). $$
In other words, \( c \) is the point in \((a, b)\) where the area under the curve is simply equal to the base length times the function height at \( c \).

We call \( f(c) = \frac{1}{b-a} \int_a^b f(t) \, dt \) the average value \( \bar{f} \) of the integrable function \( f \). That is,

\[
\bar{f} = \frac{1}{b-a} \int_a^b f(x) \, dx.
\]

Ex. The temperature during the afternoon \( t \) hours after noon is given by

\[
25 + 2t - \frac{t^2}{3}^\circ C.
\]

Find the average temperature between noon and 5p.m.
Centres of Mass:

Consider two objects located at $x_1$ and $x_2$ of mass $m_1$ and $m_2$ respectively on a stationary beam of negligible mass, as shown. Where do we put the fulcrum so that the system balances?
Let $\bar{x}$ be the point, then

$$(\bar{x} - x_1)m_1 = (x_2 - \bar{x})m_2.$$ 

Solving, we have

$$\bar{x} = \frac{x_1m_1 + x_2m_2}{m_1 + m_2}.$$ 

The quantity in the numerator is called the first moment, and the bottom is simply the total mass of the system. The point $\bar{x}$ is called the centre of mass of the system. In other words, the

$$\text{centre of mass} = \frac{\text{first moment}}{\text{total mass}}.$$
The idea generalises in the obvious way. If we have \( n \) particles with position vectors

\[ x_1, x_2, \ldots, x_n \]

with masses

\[ m_1, m_2, \ldots, m_n, \]

then the centre of mass is given by

\[ \bar{x} = \frac{m_1 x_1 + \ldots + m_n x_n}{m_1 + \ldots + m_n}. \]

This means that in many instances the system of particles may be replaced by a single particle of mass \( m_1 + \ldots + m_n \) at the point \( \bar{x} \).
Ex. In \( \mathbb{R} \), a massless beam carries a mass of 3kg, 2kg and 5kg at distances 1, 2 and 3m respectively from the left hand end. Calculate the centre of mass.
We can also consider the problem of finding the centre of mass in a continuous medium, such as a thin rod, an area or a solid. In this case, we have a continuous body (rather than a set of points) and some function (possibly constant) which gives the density of the body at any given point. We will only consider the case in $\mathbb{R}$. For higher dimensions, the problems is best solved using double and triple integrals which are covered in second year.

Consider a thin rod of length $L$ and density $\rho(x)$.  

The density is the mass per unit length, so if we take a small slice of length $\Delta x$ at a point $x$, the mass will be given by $\Delta M \approx \rho(x) \Delta x$ (mass/unit length $\times$ length). Summing and taking limits we have

$$\text{total mass} = \int_0^L \rho(x) \, dx.$$  

The $x$-moment of the element $\Delta x$ at $x$ with mass $\rho(x) \Delta x$ is given by $x \rho(x) \Delta x$, and so the total $x$-moment of the rod is given by

$$\int_0^L x \rho(x) \, dx.$$  

Thus the centre of the mass of the rod is given by

$$\bar{x} = \frac{\int_0^L x \rho(x) \, dx}{\int_0^L \rho(x) \, dx}.$$
Ex. The density of a 4 metre non-uniform metal beam, measured in kg/m, is given by 
\[ \rho(x) = 2\sqrt{x} \] where \( x \) is the distance from the left hand end of the beam. Find the total mass of the beam and the centre of mass.
Polar Co-ordinates:

As you saw in MATH1131, there are some interesting curves which can be easily expressed in polar co-ordinates but not so easily (if at all) in cartesian co-ordinates. Here we give the formulae for arc length and area in terms of polar co-ordinates.

We saw above that the arc length formula in parametric form was

$$\int_{a}^{b} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} \, dt.$$ 

If we have a polar curve in the form $r = r(\theta)$, we can put $x = r \cos \theta$ and $y = r \sin \theta$ and substitute to get

$$\text{arc length} = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta.$$
Ex. Find the arc length of the cardioid \( r = 1 + \sin \theta \).
Ex. Find the length of the spiral

\[ r = \theta^2 \]

between \( \theta = 0 \) and \( \theta = \pi \).
Area in Polar Co-ordinates:

To find the area, consider the wedge as shown in the diagram.

From high school you should know that the area of the sector is \( \frac{1}{2} r^2 \Delta \theta \). Hence, to find the area enclosed by the curve \( r = f(\theta) \), for \( \theta_1 \leq \theta \leq \theta_2 \), we calculate

\[
\frac{1}{2} \int_{\theta_1}^{\theta_2} r^2 \, d\theta.
\]
Ex. Find the area enclosed by the cardioid $r = 1 - \cos \theta$. 