Feynman's Operational Calculi: Spectral Theory for Noncommuting Self-Adjoint Operators

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Abstract. The spectral theorem for commuting self-adjoint operators along with the associated functional (or operational) calculus is among the most useful and beautiful results of analysis. It is well known that forming a functional calculus for noncommuting self-adjoint operators is far more problematic. The central result of this paper establishes a rich functional calculus for any finite number of noncommuting (i.e. not necessarily commuting) bounded, self-adjoint operators $A_1, \ldots, A_n$ and associated continuous Borel probability measures $\mu_1, \cdots, \mu_n$ on $[0,1]$. Fix $A_1, \ldots, A_n$. Then each choice of an n-tuple $(\mu_1, \ldots, \mu_n)$ of measures determines one of Feynman's operational calculi acting on a certain Banach algebra of analytic functions even when $A_1, \ldots, A_n$ are just bounded linear operators on a Banach space. The Hilbert space setting along with self-adjointness allows us to extend the operational calculi well beyond the analytic functions.

Using results and ideas drawn largely from the proof of our main theorem, we also establish a family of Trotter product type formulas suitable for Feynman's operational calculi.

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1. Introduction

Let X be a Banach space and suppose that $A_1, \ldots, A_n$ are noncommuting elements in $\mathcal{L}(X)$, the space of bounded linear operators on X. Further, for each $i \in \{1, \ldots, n\}$, let μ_i be a continuous probability measure defined on $\mathcal{B}([0,1])$, the Borel class of $[0,1]$. (Recall that a measure μ is continuous provided that $\mu(\{s\}) = 0$ for every single point set $\{s\}$.) Such measures determine an operational calculus or "disentangling map" $\mathcal{T}_{\mu_1,\ldots,\mu_n}$ from a commutative Banach algebra $\mathbb{D}(A_1, \ldots, A_n)$, called the "disentangling algebra" of analytic functions into the noncommutative Banach algebra $\mathcal{L}(X)$. (See [4] or definition 1.1 below.) It is natural to seek conditions under which such an operational calculus can be extended beyond the analytic functions in $\mathbb{D}(A_1, \ldots, A_n)$. Theorem 2.2, the main result of this paper will show, in conjunction with results from [6], that when $X = \mathcal{H}$ is a Hilbert space and $A_1, \ldots, A_n$ are self-adjoint, the domain of each of the operational calculi is much richer than $\mathbb{D}(A_1, \ldots, A_n)$.

Feynman developed 'rules' for his operational calculus for noncommuting operators while discovering the famous perturbation series and Feynman graphs of quantum electrodynamics. By the time he wrote [2] he realized that this operational calculus could be developed into a widely applicable mathematical technique. Feynman was aware that his work was far from being mathematically rigorous (see page 108 of [2]), especially with regard to the "disentangling" process, the central operation of his functional calculus. He regarded his operational calculus as a kind of generalized path integral. (See Section 14.3 of [10].)

We now give a brief description of Feynman's heuristic rules:

(a) Attach time indices to the operators to keep track of the order of the operators in products. Operators with smaller (*or earlier*) time indices are to act before operators with larger (*or later*) time indices no matter how they are *ordered* on the page.

(b) With time indices attached, functions of the operators are formed just as if they were commuting.

(c) Finally, the operator expressions are to be restored to their natural order; this is the so-called *disentangling* process. This final step is often difficult; it consists roughly of manipulating the operator expressions until their order on the page is consistent with the time ordering.

How does one accomplish (a) - (c)? There have been several quite varied approaches to this subject. Many of the references can be found in one of the books [10, 13]. We also call attention to the recent monograph by B. Jefferies [3] and the 1968 paper by M. Taylor [18]. Both of these are essentially concerned with the Weyl calculus for a finite number of noncommuting bounded operators. The paper [18] focused on operators which are also self - adjoint. The work begun by Maslov [12] and pursued by him and by several others is the furthest developed. See especially the book [13] by Nazaikinskii, Shatalov, and Sternin.

We will follow the approach initiated recently by Jefferies and Johnson ([4, 5, 6, 7]) and further developed by them, Nielsen and others ([11, 8, 14, 9]). A large family of operational calculi is defined at one time in this approach. This allows us to study a variety of operational calculi within one framework. It also permits us to solve a wide variety of evolution equations using various exponential functions of sums of noncommuting operators. (This was carried out in [8], section 4, and we hope to pursue this further in later work.) Finally, one can sometimes get information about one (or one type of) operational calculus by showing that it is the limit of simpler operational calculi. Indeed, the main theorem of this paper will rest in large part on such an argument.

Johnson and Nielsen established a stability theorem for Feynman's operational calculi [11] which will supply one of the central facts that we will need for our main result. We state that theorem now even though the precise definitions of the disentangling algebra and the disentangling map will be postponed until further on in this section.

Theorem 1.1 *For each* $i = 1, \ldots, n$, *let* μ_i *and* $\mu_{ik}, k = 1, 2, \ldots$ *be continuous probability measures on* $\mathcal{B}([0,1])$ *and suppose that the sequence* (μ_{ik}) *converges weakly to* μ_i *(denoted* $\mu_{ik} \rightharpoonup \mu_i$*) as* $k \to \infty$. *Then for every* $f \in \mathbb{D}(A_1, \ldots, A_n), \mathcal{T}_{\mu_{1k},\ldots,\mu_{nk}} f(\tilde{A}_1, \ldots, \tilde{A}_n) \to \mathcal{T}_{\mu_1,\ldots,\mu_n} f(\tilde{A}_1, \ldots, \tilde{A}_n)$ *in the operator norm on* $\mathcal{L}(X)$ *as* $k \to \infty$.

Note: (a) The weak convergence above is meant in the probabilist's sense (see [1], p. 229). (b) We can alternatively describe the conclusion of Theorem 1.1 as follows: The sequence of operational calculi specified by the sequence of n-tuples $\{(\mu_{1k}, \ldots, \mu_{nk}) : k = 1, \ldots, \infty\}$ converges as $k \to \infty$ to the operational calculus specified by the n-tuple $(\mu_1, \ldots, \mu_n)$.

We finish this introduction by briefly outlining the essential definitions and some basic facts of the approach to Feynman's operational calculi initiated in [4, 5]. A discussion of the heuristic ideas behind these operational calculi can be found in Chapter 14 of [10].

Let X be a Banach space and let $A_1, \ldots, A_n$ be nonzero bounded linear operators on X. Except for the numbers $\|A_1\|, \ldots, \|A_n\|$, which will serve as weights, we ignore for the present the nature of $A_1, \ldots, A_n$ as operators and introduce a commutative Banach algebra consisting of 'analytic functions' $f(\tilde{A}_1, \ldots, \tilde{A}_n)$, where $\tilde{A}_1, \ldots, \tilde{A}_n$ are treated as purely formal commuting objects.

Consider the collection $\mathbb{D} = \mathbb{D}(A_1, \ldots, A_n)$ of all expressions of the form

$$f(\tilde{A}_1, \ldots, \tilde{A}_n) = \sum_{m_1,\ldots,m_n=0}^{\infty} c_{m_1,\ldots,m_n} \tilde{A}_1^{m_1} \cdots \tilde{A}_n^{m_n} \tag{1.1}$$

where $c_{m_1,\dots,m_n} \in \mathbb{C}$ for all $m_1, \dots, m_n = 0, 1, \dots,$ and

$$\begin{aligned} \|f(\tilde{A}_1, \dots, \tilde{A}_n)\| &= \|f(\tilde{A}_1, \dots, \tilde{A}_n)\|_{\mathbb{D}(A_1,\dots,A_n)} \\ &:= \sum_{m_1,\dots,m_n=0}^{\infty} |c_{m_1,\dots,m_n}| \|A_1\|^{m_1} \cdots \|A_n\|^{m_n} < \infty. \end{aligned} \tag{1.2}$$

As pointed out in [4] the function on $\mathbb{D}(A_1, \dots, A_n)$ defined by (1.2) makes $\mathbb{D}(A_1, \dots, A_n)$ into a commutative Banach algebra under pointwise operations ([4], Prop. 1.1). We refer to $\mathbb{D}(A_1, \dots, A_n)$ as the *disentangling algebra* associated with the n-tuple $(A_1, \dots, A_n)$ of bounded linear operators acting on X. This commutative Banach algebra will provide us with a framework where we can apply Feynman's 'rule' (b) above rigorously rather than just heuristically.

Let $\mu_1, \dots, \mu_n$ be continuous probability measures defined at least on $\mathcal{B}([0,1])$, the Borel class of $[0,1]$. The idea is to replace the operators $A_1, \dots, A_n$ with the elements $\tilde{A}_1, \dots, \tilde{A}_n$ from $\mathbb{D}$ and then form the desired function of $\tilde{A}_1, \dots, \tilde{A}_n$. Still working in $\mathbb{D}$, we time order the expression for the function and then pass to $\mathcal{L}(X)$ simply by removing the tildes.

Given nonnegative integers $m_1, \dots, m_n$, we let $m = m_1 + \cdots + m_n$ and

$$P^{m_1,\dots,m_n}(z_1, \dots, z_n) = z_1^{m_1} \cdots z_n^{m_n}. \tag{1.3}$$

We are now ready to define the disentangling map $\mathcal{T}_{\mu_1,\dots,\mu_n}$ which will carry us from our commutative framework to the noncommutative setting of $\mathcal{L}(X)$. For $j = 1, \dots, n$ and all $s \in [0,1]$, we take $A_j(s) = A_j$ (recall that each A_j is independent of s) and, for $i = 1, \dots, m$, we define

$$C_i(s) := \begin{cases} A_1(s) & \text{if } i \in \{1, \dots, m_1\}, \\ A_2(s) & \text{if } i \in \{m_1 + 1, \dots, m_1 + m_2\}, \\ \vdots & \qquad\vdots \\ A_n(s) & \text{if } i \in \{m_1 + \cdots + m_{n-1} + 1, \dots, m\}. \end{cases} \tag{1.4}$$

For each $m = 0, 1, \dots,$ let S_m denote the set of all permutations of the integers $\{1, \dots, m\}$, and given $\pi \in S_m$, we let

$$\Delta_m(\pi) = \{(s_1, \dots, s_m) \in [0,1]^m : 0 < s_{\pi(1)} < \cdots < s_{\pi(m)} < 1\}.$$

Finally, we remark that we will use the notation μ^k to denote $\underbrace{\mu \times \cdots \times \mu}_{k\,\text{times}}$.

Definition 1.2 $\mathcal{T}_{\mu_1,\dots,\mu_n}\left(P^{m_1,\dots,m_n}(\tilde{A}_1, \dots, \tilde{A}_n)\right) :=$

$$\sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)}) (\mu_1^{m_1} \times \cdots \times \mu_n^{m_n})(ds_1, \dots, ds_m). \tag{1.5}$$

Then, for $f(\tilde{A}_1, \ldots, \tilde{A}_n) \in \mathbb{D}(A_1, \ldots, A_n)$ given by

$$f(\tilde{A}_1, \ldots, \tilde{A}_n) = \sum_{m_1, \ldots, m_n = 0}^{\infty} c_{m_1, \ldots, m_n} \tilde{A}_1^{m_1} \cdots \tilde{A}_n^{m_n}, \tag{1.6}$$

we set $\mathcal{T}_{\mu_1, \ldots, \mu_n}(f(\tilde{A}_1, \ldots, \tilde{A}_n))$ equal to

$$\sum_{m_1, \ldots, m_n = 0}^{\infty} c_{m_1, \ldots, m_n} \mathcal{T}_{\mu_1, \ldots, \mu_n} \left(P^{m_1, \ldots, m_n}(\tilde{A}_1, \ldots, \tilde{A}_n) \right). \tag{1.7}$$

Remark 1.3 *Even though the A_j's are independent of s, the* order *of the operator products in each term of equation* (1.5) *depends on the s's and on the measures $\mu_1, \ldots, \mu_n$. (If the A_j's* do *depend on s, we obtain* exactly *the same expression as seen in* (1.5) *and then we have a nontrivial integrand. But this situation will concern us only marginally in this paper. For details of the time dependent setting see, for example, the papers [8, 15, 16].)*

It is worth noting that the disentangling map as defined above is a linear operator of norm one from $\mathbb{D}(A_1, \ldots, A_n)$ to $\mathcal{L}(X)$. (See [4].) In the commutative setting, the right-hand side of (1.5) gives us $A_1^{m_1} \cdots A_n^{m_n}$, the expected result of the commutative functional calculus [4, Proposition 2.2]. (Of course, commutativity allows us to write the m operators in any desired order.) As is usual, we shall write the operator $\mathcal{T}_{\mu_1, \ldots, \mu_n} f$ in place of $\mathcal{T}_{\mu_1, \ldots, \mu_n}(f)$ for an element f of $\mathbb{D}(A_1, \ldots, A_n)$.

We shall sometimes write the bounded linear operator

$$\mathcal{T}_{\mu_1, \ldots, \mu_n}(f(\tilde{A}_1, \ldots, \tilde{A}_n))$$

as $f_{\mu_1, \ldots, \mu_n}(A_1, \ldots, A_n)$, $f_{\mu_1, \ldots, \mu_n}(\boldsymbol{A})$ with $\boldsymbol{A}$ denoting the n-tuple $(A_1, \ldots, A_n)$ of operators, or $f_{\boldsymbol{\mu}}(\boldsymbol{A})$ with $\boldsymbol{\mu}$ denoting the n–tuple $(\mu_1, \ldots, \mu_n)$ of measures. In particular,

$$P^{m_1 \ldots m_n}_{\mu_1, \ldots, \mu_n}(\boldsymbol{A}) = \mathcal{T}_{\mu_1, \ldots, \mu_n} \left(P^{m_1, \ldots, m_n}(\tilde{A}_1, \ldots, \tilde{A}_n) \right). \tag{1.8}$$

We find it convenient to use i as an index, so i denotes $\sqrt{-1}$. The real part of a complex number z is written as $\Re z$ and the imaginary part as $\Im z$. For a complex vector $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$, we set

$$\Re\zeta = (\Re\zeta_1, \ldots, \Re\zeta_n), \quad \Im\zeta = (\Im\zeta_1, \ldots, \Im\zeta_n), \quad |\zeta| = \sqrt{|\zeta_1|^2 + \cdots + |\zeta_n|^2}.$$

Remark 1.4 *A family of Trotter product type formulas suitable for Feynman's operational calculi (and mentioned in the abstract) will be established in Theorem 2.4. Here the bounded linear operators $A_1, \ldots, A_n$*

need not be self-adjoint and X can be a Banach space. However, we will require the probability measures $\mu_1, \ldots, \mu_n$ to be absolutely continuous with respect to Lebesgue measure λ.

2. The Main Theorem

We give a detailed proof that for any n-tuple $\boldsymbol{A} = (A_1, \ldots, A_n)$ of self-adjoint operators, there exists $r > 0$ such that $\boldsymbol{A}$ is of "Paley-Wiener type" $(0, r, \boldsymbol{\mu})$ for any $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$. We will follow the statement of what this means with a brief discussion of its consequences for the enlargement of the domain of the associated operational calculi.

Our main interest in this paper is in the Hilbert space setting. However, we state the definition of "Paley-Wiener type" in the more general Banach space setting.

Definition 2.1 *Let $A_1, \ldots, A_n$ be bounded linear operators acting on a Banach space X. Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ be an n-tuple of continuous probability measures on $\mathcal{B}([0,1])$ and let*

$$\mathcal{T}_{\mu_1,\ldots,\mu_n} : \mathbb{D}(A_1, \ldots, A_n) \to \mathcal{L}(X) \tag{2.1}$$

be the disentangling map defined in Definition 1.2. If there exists $C, r, s \geq 0$ such that

$$\|\mathcal{T}_{\mu_1,\ldots,\mu_n}\left(e^{\mathrm{i}(\zeta,\tilde{\boldsymbol{A}})}\right)\|_{\mathcal{L}(X)} \leq C(1+|\zeta|)^s e^{r|\Im\zeta|}, \text{ for all } \zeta \in \mathbb{C}^n, \tag{2.2}$$

then the n-tuple $\boldsymbol{A} = (A_1, \ldots, A_n)$ of operators is said to be of Paley-Wiener type $(s, r, \boldsymbol{\mu})$. (Note: Given $\boldsymbol{A}$ and $\zeta \in \mathbb{C}^n, (\zeta, \tilde{\boldsymbol{A}}) = \zeta_1\tilde{A}_1 + \cdots + \zeta_n\tilde{A}_n$.)

If the estimate (2.2) holds, then there exists a unique $\mathcal{L}(X)$-valued distribution $\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}} \in \mathcal{L}(C^\infty(\mathbb{R}^n), \mathcal{L}(X))$ such that

$$\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}(f) = (2\pi)^{-n} \int_{\mathbb{R}^n} \mathcal{T}_{\mu_1,\ldots,\mu_n}(e^{\mathrm{i}(\xi,\tilde{\boldsymbol{A}})})\hat{f}(\xi)\, d\xi, \tag{2.3}$$

for every rapidly decreasing function $f \in \mathcal{S}(\mathbb{R}^n)$. Here

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-\mathrm{i}(x,\xi)} f(x)\, dx$$

denotes the Fourier transform of f. Moreover,

$$\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}(P^{m_1,\ldots,m_n}) = P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_n}(A_1, \ldots, A_n), \tag{2.4}$$

for all nonnegative integers $m_1, \ldots, m_n$. Hence we have a rich extension of the functional calculus $f \longmapsto f_{\boldsymbol{\mu}}(\boldsymbol{A})$ from analytic functions with a uniformly convergent power series in a polydisk, to functions C^∞ in a neighbourhood of the support $\gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$ of $\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}$. In fact, all of the distributions just mentioned are compactly supported and so of finite order k. In the setting of Theorem 2.2, k will be the smallest integer strictly greater than $n/2$. (For example, $k = 2$, if $n = 2$ or 3.) Now let $K = \prod_{j=1}^n [-\|A_j\|, \|A_j\|]$. The functional calculus then extends to all functions f that are k times continuously differentiable on some open set containing K.

The support of $\gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$ of the distribution $\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}$ is defined as the μ–joint spectrum of the n–tuple $\boldsymbol{A} = (A_1, \ldots, A_n)$. The distribution $\mathcal{F}_{\boldsymbol{\mu},\boldsymbol{A}}$ is called Feynman's $\boldsymbol{\mu}$–functional calculus for $\boldsymbol{A}$. The number $r_{\boldsymbol{\mu}}(\boldsymbol{A}) = \sup\{|x| : x \in \gamma_{\boldsymbol{\mu}}(\boldsymbol{A})\}$ is called the μ–joint spectral radius of $\boldsymbol{A}$. It is shown in [7] via Clifford analysis that the nonempty compact subset $\gamma_{\boldsymbol{\mu}}(\boldsymbol{A})$ of $\mathbb{R}^n$ may be interpreted as the set of singularities of a multidimensional analogue of the resolvent family of a single operator. For more detail on or related to the last two paragraphs, see pages 186-192 and especially Theorem 3.1 and Proposition 3.2 of [6].

If we have more information about the particular operators and measures that are involved, we can sometimes further enlarge the functional calculus. Example 2.1, p.176-178 in [6] is an extreme case. Here A_1 and A_2 are the 2 by 2, self-adjoint Pauli matrices σ_1 and σ_3. The measures μ_1 and μ_2 are any continuous probability measures with the support of μ_1 entirely to the left of the support of μ_2. In this case, $f_{\mu_1,\mu_2}(A_1, A_2)$ makes sense for any f which is defined on the 4 point set $\{-1, 1\} \times \{-1, 1\}$. This last set is the product of the ordinary spectrums of σ_1 and σ_3.

Theorem 2.2 *An n-tuple* $\boldsymbol{A} = (A_1, \ldots, A_n)$ *of bounded self-adjoint operators acting on a Hilbert space H is of Paley-Wiener type* $(0, r, \boldsymbol{\mu})$ *with* $r = (\|A_1\|^2 + \cdots + \|A_n\|^2)^{1/2}$, *for any n-tuple* $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ *of continuous probability measures on* $\mathcal{B}([0, 1])$.

Proof One of the keys to the proof is a use of the Martingale Convergence Theorem. We can apply it to the Radon-Nikodyn derivative of probability measures on $[0, 1]$ which are absolutely continuous with respect to Lebesgue measure λ. Such Radon-Nikodyn derivatives are nonnegative functions with $L^1(\lambda)$-norm 1. However, we are only assuming that $\mu_1, \ldots, \mu_n$ are continuous and so, given μ_i, we will begin by finding a sequence of absolutely continuous probability measures which converge weakly to μ_i. (We will, but do not need to, do this even for the μ_i's that are absolutely continuous with respect to λ.)

Let μ be a continuous Borel probability measure on $[0,1]$. Let $\rho : [0,1] \to \mathbb{R}$ be a nonnegative continuous function with compact support in $[0,1)$ and $\|\rho\|_1 = 1$. It will be convenient to let $\rho(x) = 0$ for $x < 0$. For every $\epsilon \in (0,1]$, we set $\rho_\epsilon(x) = \epsilon^{-1}\rho(x/\epsilon), 0 \le x \le 1$. It is easy to check that $\|\rho_\epsilon\|_1 = 1$ for $0 < \epsilon \le 1$. Now we let

$$(\rho_\epsilon * \mu)(x) := \int_0^x \rho_\epsilon(x-y)\mu(dy),\ 0 \le x \le 1. \tag{2.5}$$

We assert that $(\rho_\epsilon * \mu)\lambda \rightharpoonup \mu$ as $\epsilon \to 0^+$. Indeed, using the definition of weak convergence, our assertion follows once we know that, for every continuous function $\phi : [0,1] \to \mathbb{R}$,

$$\int_0^1 (\rho_\epsilon * \mu)(x)\phi(x)dx \to \int_0^1 \phi(x)\mu(dx) \text{ as } \epsilon \to 0^+. \tag{2.6}$$

We omit the proof of the limit (2.6) as it can be carried out using standard techniques. The basic idea of the proof is much like arguments using approximate identities although some of the particular details follow Exercise 10, p. 194 of [17] more closely.

Now consider the partition of the interval $[0,1)$ into n^k disjoint intervals $I_{k,\ell} = [(\ell-1)n^{-k}, \ell n^{-k}),\ \ell = 1, \ldots, n^k$ each of length n^{-k}. (Below, n will be the number of operator-measure pairs in our problem.) The collection $\mathcal{A}_k$ of finite disjoint unions of these intervals is an algebra – in fact a σ-algebra. Note that $\mathcal{A}_k \subset \mathcal{A}_{k+1}$ for $k = 1, 2, \ldots$. Let $P_k : L^1(\lambda) \to L^1(\lambda)$ be the conditional expectation operator [1, p.265] with respect to $\mathcal{A}_k$; that is,

$$P_k f = n^k \sum_{\ell=1}^{n^k} \chi_{I_{k,\ell}} \int_{I_{k,\ell}} f\, d\lambda. \tag{2.7}$$

Note that the sequence $\{P_k f\}$ is adapted with respect to the sequence $\{\mathcal{A}_k\}$ of σ-algebras [1, p.280]. Then for each $f \in L^1(\lambda)$, by the Martingale Convergence Theorem [1, p.285-6], $P_k f \to f$ in $L^1(\lambda)$ (and λ-a.e.) as $k \to \infty$. Further, by Theorem 10.1.3 from [1], we have

$$\int_0^1 P_k f\ d\lambda = \int_0^1 f d\lambda,\ k = 1, 2, \ldots. \tag{2.8}$$

(Remark: The usual notation for $P_k f$ in the probability literature is $\mathbb{E}(f|\mathcal{A}_k)$.)

Now set $f_{i,j,k} := P_k(\rho_{1/j} * \mu_i)$ for each $i = 1, \ldots, n$ and $j, k = 1, 2, \ldots$. Each such step function $f_{i,j,k}$ is constant on each interval $I_{k,\ell}, \ell = 1, \ldots, n^k$ and

$$f_{i,j,k} := n^k \sum_{\ell=1}^{n^k} \chi_{I_{k,\ell}} \int_{I_{k,\ell}} \rho_{1/j} * \mu_i\, d\lambda. \tag{2.9}$$

The function $\tilde{f}_{i,j,k}$ that we are about to define is a key to the proof:

$$\tilde{f}_{i,j,k} := n^k \sum_{m=0}^{n^{k-1}-1} \chi_{I_{k,mn+i}} \int_{I_{k-1,m+1}} \rho_{1/j} * \mu_i \, d\lambda. \tag{2.10}$$

Note that this function has support in the finite union

$$J_{i,k} := \bigcup_{m=0}^{n^{k-1}-1} I_{k,mn+i}$$

of disjoint intervals, for each $i = 1, \ldots, n$ and $J_{i,k} \cap J_{\ell,k} = \emptyset$ for $i \neq \ell$. The integral $\int_{I_{k-1,m+1}} \rho_{1/j} * \mu_i \, d\lambda$ in the sum (2.10) is a weight factor compensating for the omission of terms from the sum (2.9), so that the following equalities hold true:

$$\int_0^1 f_{i,j,k} \, d\lambda = \int_0^1 \rho_{1/j} * \mu_i \, d\lambda = \int_0^1 \tilde{f}_{i,j,k} \, d\lambda. \tag{2.11}$$

The first equality follows from (2.7), (2.8) and the definition of $f_{i,j,k}$ above.

We turn now to the second equality in (2.11):

$$\begin{aligned}\int_0^1 \tilde{f}_{i,j,k} \, d\lambda &= n^k \sum_{m=0}^{n^{k-1}-1} \frac{1}{n^k} \int_{I_{k-1,m+1}} \rho_{1/j} * \mu_i \, d\lambda \\ &= \int_{I_{k-1,1}} \rho_{1/j} * \mu_i \, d\lambda + \int_{I_{k-1,2}} \rho_{1/j} * \mu_i \, d\lambda + \ldots \\ &\quad + \int_{I_{k-1,n^{k-1}}} \rho_{1/j} * \mu_k \, d\lambda \\ &= \int_0^1 \rho_{1/j} * \mu_i \, d\lambda.\end{aligned} \tag{2.12}$$

Thus (2.11) is established.

Now let ϕ be a continuous function on $[0,1]$. Given $\epsilon > 0$, use the uniform continuity of ϕ to choose k so large that $|\phi(x) - \phi(y)| < \epsilon$ whenever $|x-y| < n^{-k+1}$. We wish to compare the integrals $\int_0^1 f_{i,j,k}\phi \, d\lambda$ and $\int_0^1 \tilde{f}_{i,j,k}\phi \, d\lambda$. We begin with the first of these. The RHS of the 2nd equality in (2.13) below is just another way of writing the sum of the

n^k terms that appear on the LHS.

$$\begin{aligned}\int_0^1 f_{i,j,k}\phi\, d\lambda &= \int_0^1 \left\{ n^k \sum_{p=1}^{n^k} \chi_{I_{k,p}}\phi \int_{I_{k,p}} \rho_{1/j} * \mu_i \, d\lambda \right\} d\lambda \\ &= \int_0^1 n^k \left\{ \sum_{m=0}^{n^{k-1}-1} \sum_{\ell=1}^{n} \chi_{I_{k,mn+\ell}}\phi \left(\int_{I_{k,mn+\ell}} \rho_{1/j} * \mu_i \, d\lambda \right) \right\} d\lambda \quad (2.13) \\ &= n^k \sum_{m=0}^{n^{k-1}-1} \sum_{\ell=1}^{n} \left(\int_{I_{k,mn+\ell}} \rho_{1/j} * \mu_i \, d\lambda \right) \left(\int_{I_{k,mn+\ell}} \phi \, d\lambda \right).\end{aligned}$$

On the other hand, starting with (2.10) we have

$$\begin{aligned}\int_0^1 \tilde{f}_{i,j,k}\phi\, d\lambda &= \int_0^1 n^k \sum_{m=0}^{n^{k-1}-1} \chi_{I_{k,mn+i}}\phi \left(\int_{I_{k-1,m+1}} \rho_{1/j} * \mu_i \, d\lambda \right) d\lambda \\ &= n^k \sum_{m=0}^{n^{k-1}-1} \left(\int_{I_{k-1,m+1}} \rho_{1/j} * \mu_i \, d\lambda \right) \left(\int_{I_{k,mn+i}} \phi d\lambda \right) \quad (2.14) \\ &= n^k \sum_{m=0}^{n^{k-1}-1} \sum_{\ell=1}^{n} \left(\int_{I_{k,mn+\ell}} \rho_{1/j} * \mu_i \, d\lambda \right) \left(\int_{I_{k,mn+i}} \phi \, d\lambda \right).\end{aligned}$$

First using (2.13) and (2.14) and then the Mean Value Theorem for Integrals, there exist $\xi_{k,\ell} \in I_{k,\ell}, \ell = 1, \ldots, n^k$ such that

$$\begin{aligned}&\left| \int_0^1 \left(f_{i,j,k} - \tilde{f}_{i,j,k} \right) \phi \, d\lambda \right| \quad (2.15) \\ &\leq \sum_{m=0}^{n^{k-1}-1} \sum_{\ell=1}^{n} \left(\int_{I_{k,mn+\ell}} \rho_{1/j} * \mu_i \, d\lambda \right) \left| n^k \int_{I_{k,mn+\ell}} \phi \, d\lambda - n^k \int_{I_{k,mn+i}} \phi \, d\lambda \right| \\ &= \sum_{m=0}^{n^{k-1}-1} \sum_{\ell=1}^{n} \int_{I_{k,mn+\ell}} \rho_{1/j} * \mu_i \, d\lambda \left| \phi\left(\xi_{k,mn+\ell}\right) - \phi\left(\xi_{k,mn+i}\right) \right| < \epsilon.\end{aligned}$$

The final inequality follows from the fact that $|\xi_{k,mn+\ell} - \xi_{k,mn+i}| < n^{-k+1}$ for each $i \in \{1, \ldots, n\}$ and each $j \in \{1, 2, 3, \ldots\}$. Thus for each $i \in \{1, \ldots, n\}$ and each $j \in \{1, \ldots, m\}$,

$$f_{i,j,k}\lambda - \tilde{f}_{i,j,k}\lambda \rightharpoonup 0 \text{ as } k \to \infty. \quad (2.16)$$

Let $\mu_{i,j,k} := f_{i,j,k}\lambda, i = 1, \ldots, n$ and $j, k = 1, 2, \ldots$.

Next we summarize what we have proved so far about the weak limits and then draw some conclusions. For each $i = 1, \ldots, n$, the following limits obtain.

A $\left(\rho_{1/j} * \mu_i\right) \lambda \rightharpoonup \mu_i$ as $j \to \infty$.

B $f_{i,j,k}\lambda = P_k \left(\rho_{1/j} * \mu_i\right) \lambda \rightharpoonup \left(\rho_{1/j} * \mu_i\right) \lambda$ as $k \to \infty$.

In fact, the probability densities in **B** converge in L^1-norm and so the measures converge in total variation norm and so certainly converge weakly.

C $f_{i,j,k}\lambda - \tilde{f}_{i,j,k}\lambda \rightharpoonup 0$ as $k \to \infty$.

From B and C we see that

D $f_{i,j,k}\lambda = P_k \left(\rho_{1/j} * \mu_i\right) \lambda \rightharpoonup \left(\rho_{1/j} * \mu_i\right) \lambda$ as $k \to \infty$.

Now from D and A we have the iterated weak limits,

E $\lim_{j\to\infty} \left[\lim_{k\to\infty} \left(\tilde{f}_{i,j,k}\right) \lambda\right] = \lim_{j\to\infty} \rho_{1/j} * \mu_i = \mu_i$.

Since $[0, 1]$ is a separable metric space, it follows from E that there exists a sequence $(\tilde{f}_{i,j,k(j)})\lambda$ such that for each $i = 1, \ldots, n$,

F $\left(\tilde{f}_{i,j,k(j)}\right) \lambda \rightharpoonup \mu_i$ as $j \to \infty$.

(See [1, p. 309-10] and especially the paragraph preceding Theorem 11.3.3.) Note that the increasing sequence $k(j)$, $j = 1, 2, \ldots,$ may be chosen independently of $i = 1, \ldots, n$ by observing that the space $\mathcal{M}([0, 1], \mathbb{R}^n)$ of $\mathbb{R}^n$-valued Borel measures on $[0, 1]$ is in duality with the space $C([0, 1], \mathbb{R}^n)$ of $\mathbb{R}^n$-valued continuous functions and that closed balls in $\mathcal{M}([0, 1], \mathbb{R}^n)$ are compact and metrizable for the associated weak*-topology.

Note: As we continue we will just write j, k but will understand that $k = k(j)$ as in F above.

The measure $\mu_{i,j,k}$ is given by

$$\mu_{i,j,k} = \tilde{f}_{i,j,k}\lambda = n^k \sum_{m=0}^{n^{k-1}-1} \left(\chi_{I_{k,mn+i}}\lambda\right) \int_{I_{k-1,m+1}} \rho_{1/j} * \mu_i \, d\lambda. \quad (2.17)$$

Before beginning the calculation below we make some simple comments and introduce some notation.

(a) Since exponential functions are entire, the exponential functions involved below are certainly in the domain of the disentangling map.

(b) We will write $(\zeta, \boldsymbol{A}) = \zeta_1 A_1 + \cdots + \zeta_n A_n$ where $\zeta = (\zeta_1, \ldots, \zeta_n)$ is an n-tuple of complex numbers and $\boldsymbol{A} = (A_1, \ldots, A_n)$. Similar notation will be used in connection with $\tilde{\boldsymbol{A}}$.

(c) Since the disentangling algebra is commutative, we have

$$e^{(\zeta, \tilde{\boldsymbol{A}})} = e^{\zeta_1 \tilde{A}_1 + \cdots + \zeta_n \tilde{A}_n} = e^{\zeta_1 \tilde{A}_1} \ldots e^{\zeta_n \tilde{A}_n}.$$

Because of how the function $\tilde{f}_{i,j,k}$ is supported (see equation (2.10)) and since $\mu_{i,j,k}$ is defined using $\tilde{f}_{i,j,k}$, an extension of Proposition 2.2 from [6] allows us to do the following calculation (much as was done in Example 2.2 of that paper):

$$\begin{aligned}
&\mathcal{T}_{\mu_{1,j,k}, \ldots, \mu_{n,j,k}}(e^{\mathrm{i}(\zeta, \tilde{\boldsymbol{A}})}) \\
&= \Bigg(\exp\left\{ \mathrm{i}\zeta_n \left(\int_{I_{k-1,n^{k-1}}} \rho_{1/j} * \mu_n d\lambda \right) A_n \right\} \\
&\ldots \exp\left\{ \mathrm{i}\zeta_1 \left(\int_{I_{k-1,n^{k-1}}} \rho_{1/j} * \mu_1 d\lambda \right) A_1 \right\} \Bigg) \ldots \\
&\Bigg(\exp\left\{ \mathrm{i}\zeta_n \left(\int_{I_{k-1,2}} \rho_{1/j} * \mu_n d\lambda \right) A_n \right\} \ldots \\
&\qquad \exp\left\{ \mathrm{i}\zeta_1 \left(\int_{I_{k-1,2}} \rho_{1/j} * \mu_1 d\lambda \right) A_1 \right\} \Bigg). \\
&\Bigg(\exp\left\{ \mathrm{i}\zeta_n \left(\int_{I_{k-1,1}} \rho_{1/j} * \mu_n d\lambda \right) A_n \right\} \ldots \\
&\qquad \exp\left\{ \mathrm{i}\zeta_1 \left(\int_{I_{k-1,1}} \rho_{1/j} * \mu_1 d\lambda \right) A_1 \right\} \Bigg).
\end{aligned} \tag{2.18}$$

Breaking $\zeta_1 = \xi_1 + \mathrm{i}\eta_1$ into real and imaginary parts, we have $\mathrm{i}\zeta_1 = -\eta_1 + \mathrm{i}\xi_1$. Further, $|e^{\mathrm{i}\zeta_1}| = |e^{\mathrm{i}\xi_1}||e^{-\eta_1}| \leq e^{|\eta_1|} = e^{|\Im\zeta_1|}$. Similarly, $|e^{\mathrm{i}\zeta_2}| \leq e^{|\Im\zeta_2|}, \ldots, |e^{\mathrm{i}\zeta_n}| \leq e^{|\Im\zeta_n|}$.

Now using (2.18), the selfadjointness of A_i, $i = 1, \ldots, n$, the standard multiplicative Banach algebra inequality and the simple computations just above, we can write

$$\begin{aligned}
&\|\mathcal{T}_{\mu_{1,j,k},\ldots,\mu_{n,j,k}}(e^{\mathrm{i}(\zeta,\tilde{\boldsymbol{A}})})\| \\
&\le \|\exp\left\{\mathrm{i}\zeta_n\left(\int_{I_{k-1,n^{k-1}}}\rho_{1/j}*\mu_n d\lambda\right)A_n\right\}\|\ldots \\
&\qquad \|\exp\left\{\mathrm{i}\zeta_1\left(\int_{I_{k-1,n^{k-1}}}\rho_{1/j}*\mu_1 d\lambda\right)A_1\right\}\|\ldots \\
&\|\exp\left\{\mathrm{i}\zeta_n\left(\int_{I_{k-1,2}}\rho_{1/j}*\mu_n d\lambda\right)A_n\right\}\|\ldots \qquad (2.19)\\
&\qquad \|\exp\left\{\mathrm{i}\zeta_1\left(\int_{I_{k-1,2}}\rho_{1/j}*\mu_1 d\lambda\right)A_1\right\}\|. \\
&\|\exp\left\{\mathrm{i}\zeta_n\left(\int_{I_{k-1,1}}\rho_{1/j}*\mu_n d\lambda\right)A_n\right\}\|\ldots \\
&\qquad \|\exp\left\{\mathrm{i}\zeta_1\left(\int_{I_{k-1,1}}\rho_{1/j}*\mu_1 d\lambda\right)A_1\right\}\|
\end{aligned}$$

$$\begin{aligned}
&\le e^{|\Im\zeta_n|\left(\int_{I_{k-1,n^{k-1}}}\rho_{1/j}*\mu_n d\lambda\right)\|A_n\|}\ldots e^{|\Im\zeta_1|\left(\int_{I_{k-1,n^{k-1}}}\rho_{1/j}*\mu_1 d\lambda\right)\|A_1\|}\ldots \\
&\quad e^{|\Im\zeta_n|\left(\int_{I_{k-1,2}}\rho_{1/j}*\mu_n d\lambda\right)\|A_n\|}\ldots e^{|\Im\zeta_1|\left(\int_{I_{k-1,2}}\rho_{1/j}*\mu_1 d\lambda\right)\|A_1\|}. \\
&\quad e^{|\Im\zeta_n|\left(\int_{I_{k-1,1}}\rho_{1/j}*\mu_n d\lambda\right)\|A_n\|}\ldots e^{|\Im\zeta_1|\left(\int_{I_{k-1,1}}\rho_{1/j}*\mu_1 d\lambda\right)\|A_1\|}.
\end{aligned}$$

Now we are dealing with numbers and so the noncommutativity of the operators is not an issue. Since

$$\begin{aligned}
&\int_{I_{k-1,1}}\rho_{1/j}*\mu_1 d\lambda+\int_{I_{k-1,2}}\rho_{1/j}*\mu_1 d\lambda+\ldots \\
&+\int_{I_{k-1,n^{k-1}}}\rho_{1/j}*\mu_1 d\lambda=\int_0^1\rho_{1/j}*\mu_1 d\lambda=1,
\end{aligned}$$

the product of the terms in the column furthest to the right is $e^{|\Im\zeta_1|\|A_1\|}$.

A similar thing happens in the other columns and so we obtain

$$\begin{aligned}
&\|\mathcal{T}_{\mu_{1,j,k},\ldots,\mu_{n,j,k}}(e^{\mathrm{i}(\zeta,\tilde{A})})\|\le e^{(|\Im\zeta_1|\|A_1\|+|\Im\zeta_2\|A_2\|+\cdots+|\Im\zeta_n|\|A_n\|)} \\
&\le e^{[|\Im\zeta_1|^2+\cdots+|\Im\zeta_n|^2]^{1/2}[\|A_1\|^2+\cdots+\|A_n\|^2]^{1/2}}=e^{r|\Im\zeta|}
\end{aligned} \tag{2.20}$$

where $r=[\|A_1\|^2+\cdots+\|A_n\|^2]^{1/2}$ and $|\Im\zeta|$ is the Euclidean norm in $\mathbb{R}^n$ of the vector $(\Im\zeta_1,\ldots,\Im\zeta_n)$.

Theorem 1.1 and the inequality (2.20) show that the bounded self-adjoint operators $A_1, \ldots, A_n$ are of Paley-Wiener type $(0, r, \boldsymbol{\mu})$ where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$. □

Remark 2.3 *Following a suggestion of Jefferies, Johnson and Nielsen used the main theorem from [11], that is, Theorem 1.1 of this paper, to prove the special case of Theorem 2.2 where $n = 2$ and the operational calculus is the Weyl calculus. In the closing remark of [11], Johnson and Nielsen asserted that an argument similar to the one in that paper would take care of the case of the Weyl calculus for any n. In fact, the argument in [11] fails at a critical point for $n \geq 3$.*

The following result is a simple consequence of Theorem 1.1 and the proof of Theorem 2.2.

Corollary 2.1 *Let $A_1, \ldots, A_n$ be bounded, self-adjoint operators on the Hilbert space H and let $\mu_1, \ldots, \mu_n$ be absolutely continuous probability measures on $\mathcal{B}([0,1])$. Finally let $\xi = (\xi_1, \ldots \xi_n)$ be an n-tuple of real numbers. Then*

$$\|\mathcal{T}_{\mu_1,\ldots,\mu_n}(e^{\mathrm{i}(\xi,\tilde{\boldsymbol{A}})})\| = \|\mathcal{T}_{\mu_1,\ldots\mu_n}(e^{\mathrm{i}\xi_1\tilde{A}_1} \cdots e^{\mathrm{i}\xi_n\tilde{A}_n})\| = 1. \tag{2.21}$$

Proof. Let ζ from Theorem 2.2 (see also Definition 2.1) equal ξ as above. Then we see that the operator on the RHS of (2.18) is unitary and so has norm 1. But, by Theorem 1.1 and the fact that $\mu_{\mathrm{i},j,k} \rightharpoonup \mu_i$ for each $i = 1, \ldots, n$, we have the operator norm convergence of the squence of operators in (2.18) to $\mathcal{T}_{\mu_1,\ldots,\mu_n}(e^{\mathrm{i}(\xi,\tilde{A})})$. The equality in (2.21) follows. □

The reader may have noticed that we did not need the self-adjointness of the operators nor even the Hilbert space setting until late in the proof of Theorem 2.2. In fact, we can use some of the early parts of the proof to establish a class of "Trotter product formulas" suitable for Feynman's operational calculi in the general Banach space setting.

Theorem 2.4 *Let X be a Banach space over $\mathbb{C}$ and let $\zeta = (\zeta_1, \ldots, \zeta_n)$ be an n-tuple of complex numbers. Further, let $\mu_1, \ldots, \mu_n$ be probability measures on $\mathcal{B}([0,1])$ each of which is absolutely continuous with respect to λ; hence, $\mu_1 = g_1\lambda, \ldots, \mu_n = g_n\lambda$ where $g_1, \ldots, g_n$ are nonnegative functions in $L^1([0,1], \mathcal{B}([0,1]), \lambda)$. Finally, let $A = (A_1, \ldots, A_n)$ be an n-tuple of bounded linear operators on X. Then*

$$\|\mathcal{T}_{\mu_1,\ldots,\mu_n}(e^{\mathrm{i}(\zeta,\tilde{\boldsymbol{A}})}) - \mathcal{T}_{\mu_{1,k},\ldots,\mu_{n,k}}(e^{\mathrm{i}(\zeta,\tilde{\boldsymbol{A}})})\|_{\mathcal{L}(X)} \to 0 \tag{2.22}$$

as $k \to \infty$ *where*

$$\mathcal{T}_{\mu_{1,k},\dots,\mu_{n,k}}(e^{\mathrm{i}(\zeta,\tilde{\boldsymbol{A}})}) = \mathcal{T}_{\mu_{1,k},\dots,\mu_{n,k}}(e^{\mathrm{i}\zeta_1\tilde{A}_1}\dots e^{\mathrm{i}\zeta_n\tilde{A}_n})$$
$$= \exp\left\{\mathrm{i}\zeta_n\left(\int_{I_{k-1,n^{k-1}}} g_n d\lambda\right)A_n\right\}\dots\exp\left\{\mathrm{i}\zeta_1\left(\int_{I_{k-1,n^{k-1}}} g_1 d\lambda\right)A_1\right\}\dots$$
$$\exp\left\{\mathrm{i}\zeta_n\left(\int_{I_{k-1,2}} g_n d\lambda\right)A_n\right\}\dots\exp\left\{\mathrm{i}\zeta_1\left(\int_{I_{k-1,2}} g_1 d\lambda\right)A_1\right\}$$
$$\exp\left\{\mathrm{i}\zeta_n\left(\int_{I_{k-1,1}} g_n d\lambda\right)A_n\right\}\dots\exp\left\{\mathrm{i}\zeta_1\left(\int_{I_{k-1,1}} g_1 d\lambda\right)A_1\right\}. \tag{2.23}$$

Proof. We will just comment on which parts of the proof of Theorem 2.2 are needed here and which are not. We will also note the point at which Theorem 1.1 is applied.

The first part of the proof of Theorem 2.2 is needed only for measures which are continuous but not absolutely continuous. We have no such measures here. The second part of the earlier proof involves the Martingale Convergence Theorem. We need this and we obtain $f_{\mathrm{i},k} = P_k g_i, i = 1,\dots,n$ with (2.9) suitably adjusted. (Note that we do not need the index j as our measures are all absolutely continuous.) The function $\tilde{f}_{\mathrm{i},k}$ is then defined with the integrands in (2.10) changed to g_i. The proof now moves along with no j's involved and with $\rho_{1/j} * \mu_i$ replaced by $g_i, i = 1,\dots,n$.

When we reach the summary A-F, A is not needed. Replace B with $f_{i,k}\lambda = (P_k g_i)\lambda \rightharpoonup g_i\lambda$ as $k \to \infty$ for $i = 1,\dots,n$. Change C to $f_{i,k}\lambda - \tilde{f}_{i,k}\lambda \rightharpoonup 0$ as $k \to \infty$ for $i = 1,\dots,n$. Based on the revised form of B and C, change D to $\tilde{f}_{i,k}\lambda \rightharpoonup g_i\lambda$ as $k \to \infty$; that is

$$\mu_{i,k} \rightharpoonup \mu_i$$

as $k \to \infty$ for $i = 1,\dots,n$. (Note that in (2.17) $\mu_{i,j,k}, \tilde{f}_{i,j,k}$ and $\rho_{1/j} * \mu_i$ are replaced by $\mu_{i,k}, \tilde{f}_{i,k}$ and g_i, respectively.)

The calculation in (2.18) is done in the same way but the subscript j on the LHS is missing and $\rho_{1/j} * \mu_i$ is replaced by $g_i, i = 1,\dots,n$, on the RHS.

We can now apply Theorem 1.1 to finish the proof of this theorem. □

Remark 2.5 *(a) The inequalities which concerned us toward the end of the proof of Theorem 2.2 did not concern us in the proof of Theorem 2.4 since we were not trying to show that* $(A_1,\dots,A_n)$ *is of Paley-Wiener type* $(0, r, (\mu_1,\dots,\mu_n))$*.*

(b) The Trotter products on the RHS of (2.23) look more like the usual Trotter products when special choices are made for ζ. Some examples: (i) Each ζ_j equals i *(or t*i*). (ii) Each ζ_j equals -1 (or $-t$).*

(c) We hope to investigate variations and consequences of Theorem 2.4 in later work.

Remark 2.6 *In view of Corollary 3.1 of the paper [14], we may write equation (2.22) as*

$$\|\mathcal{T}_{\lambda,\ldots,\lambda}(e^{\mathrm{i}(\zeta,\widetilde{[\mathbf{g}\cdot\mathbf{A}]})}) - \mathcal{T}_{\mu_{1,k},\ldots,\mu_{n,k}}(e^{\mathrm{i}(\zeta,\tilde{\mathbf{A}})})\|_{\mathcal{L}(X)} \to 0 \tag{2.24}$$

where $\widetilde{[\mathbf{g}\cdot\mathbf{A}]} := \left(\widetilde{[g_1 \cdot A_1]}, \ldots, \widetilde{[g_n \cdot A_n]}\right)$, that is the time independent operators $A_1, \ldots, A_n$ are replaced by the time dependent operators $g_1 \cdot A_1, \ldots, g_n \cdot A_n$ where $g_1 = \frac{d\mu_1}{d\lambda}, \ldots, g_n = \frac{d\mu_n}{d\lambda}$.

We now present two simple examples illustrating Theorem 2.4.

Example 2.7 *For the first example we assume that, in Theorem 2.4, $g_1 = \cdots = g_n = 1$; i.e. Lebesgue measure is associated to each operator. It follows from [5, Lemma 5.4] that*

$$\mathcal{T}_{\mu_1,\ldots,\mu_n}(e^{\mathrm{i}(\zeta,\tilde{\boldsymbol{A}})}) = \mathcal{T}_{\lambda,\ldots,\lambda}(e^{\mathrm{i}(\zeta,\tilde{\boldsymbol{A}})}) = e^{\mathrm{i}(\zeta,\boldsymbol{A})} \tag{2.25}$$

and so Theorem 2.4 tells us that

$$e^{\mathrm{i}(\zeta,\boldsymbol{A})} = \lim_{k\to\infty}\left\{\exp\left\{\frac{\zeta_n}{n^{k-1}}A_n\right\}\ldots\exp\left\{\frac{\zeta_1}{n^{k-1}}A_1\right\}\right\}^{n^{k-1}} \tag{2.26}$$

Example 2.8 *In the second example, we will consider two operators, A_1 and A_2 and we will take $\mu_1 = 2t\,d\lambda$ and $\mu_2 = 3t^2\,d\lambda$. For any nonnegative integer l, we have*

$$\int_{I_{k-1,l}} g_1\,d\lambda = \frac{2l-1}{2^{2k-2}}, \tag{2.27}$$

and

$$\int_{I_{k-1,l}} g_2\,d\lambda = \frac{3l^2-3l+1}{2^{3k-3}}. \tag{2.28}$$

Theorem 2.4 tells us that

$$\begin{aligned}
&\lim_{k\to\infty} \mathcal{T}_{\mu_{1,k},\mu_{2,k}} e^{\mathrm{i}(\zeta_1\tilde{A}_1+\zeta_2\tilde{A}_2)} \\
&= \lim_{k\to\infty} \left\{ \exp\left(\mathrm{i}\zeta_2 \left[\frac{3\cdot 2^{2k-2}-3\cdot 2^{k-1}+1}{2^{3k-3}}\right] A_2\right)\cdot \right. \\
&\quad \exp\left(\mathrm{i}\zeta_1 \left[\frac{2\cdot 2^{k-1}-1}{2^{2k-2}}\right] A_1\right)\cdots \\
&\quad \exp\left(\mathrm{i}\zeta_2 \left[\frac{7}{2^{3k-3}}\right] A_2\right) \exp\left(\mathrm{i}\zeta_1 \left[\frac{3}{2^{2k-2}}\right] A_1\right)\cdot \\
&\quad \left. \exp\left(\mathrm{i}\zeta_2 \left[\frac{1}{2^{3k-3}}\right] A_2\right) \exp\left(\mathrm{i}\zeta_1 \left[\frac{1}{2^{2k-2}}\right] A_1\right)\right\} \\
&= \mathcal{T}_{\mu_1,\mu_2} e^{\mathrm{i}(\zeta_1\tilde{A}_1+\zeta_2\tilde{A}_2)}.
\end{aligned} \tag{2.29}$$

Using Corollary 3.1 of [14], the last expression immediately above can be written as $\mathcal{T}_{\lambda,\lambda} e^{\mathrm{i}(\zeta_1[\widetilde{g_1\cdot A_1}]+\zeta_2[\widetilde{g_2\cdot A_2}])}$ *where the time dependence is carried by the functions* g_1 *and* g_2*.*

It is clear from Example 2.8 that we can produce an infinite number of distinct Trotter product formulas by varying the probability densities g_1 and g_2.

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