Feynman’s Operational Calculi: Decomposing Disentanglings

B. Jefferies · G.W. Johnson

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Abstract Let $X$ be a Banach space and suppose that $A_1, \ldots, A_n$ are noncommuting (that is, not necessarily commuting) elements in $L(X)$, the space of bounded linear operators on $X$. Further, for each $i \in \{1, \ldots, n\}$, let $\mu_i$ be a continuous probability measure on $B([0, 1])$, the Borel class of $[0, 1]$. Each such $n$-tuple of operator-measure pairs $(A_i, \mu_i), i = 1, \ldots, n$, determines an operational calculus or disentangling map $T_{\mu_1, \ldots, \mu_n}$ from a commutative Banach algebra $\mathbb{D}(A_1, \ldots, A_n)$ of analytic functions, called the disentangling algebra, into the noncommutative Banach algebra $L(X)$. The disentanglings are the central processes of Feynman’s operational calculi.

We partition the interval $[0, 1]$ and show in this paper how the disentangling over $[0, 1]$ can be decomposed into the disentanglings over the subintervals associated with the partition. This often enables us to simplify the disentangling process and, in some cases, to calculate it completely. This includes circumstances we could not previously deal with.

One of the major motivations for developing these operational calculi is for representing various evolutions. It is natural to ask how a disentangled exponential $T_{\mu_1, \ldots, \mu_n}(e^{\hat{A}_1 + \cdots + \hat{A}_n})$ behaves when $[0, 1]$ is partitioned into disjoint subintervals and we decompose the indicated disentangling. A corollary of the main theorem of this paper resolves this general question. (The main theorem itself has corollaries which are by no means limited to exponential functions.)

Keywords Feynman’s operational calculus · Functional calculus · Disentangling

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1 Introduction

Feynman’s operational calculus for an \( n \)-tuple of operator-measure pairs \( (A_i, \mu_i), \ i = 1, \ldots, n \), prescribes possible functions \( f_{\mu_1, \ldots, \mu_n}(A_1, \ldots, A_n) \) of the operators \( A_1, \ldots, A_n \) by means of the measures \( \mu_1, \ldots, \mu_n \) weighting all possible choices of operator products. The disentangling map \( T_{\mu_1, \ldots, \mu_n} : f \mapsto f_{\mu_1, \ldots, \mu_n}(A_1, \ldots, A_n) \) maps elements of the commutative disentangling algebra \( \mathbb{D}(A_1, \ldots, A_n) \) into the noncommutative Banach algebra \( \mathcal{L}(X) \).

Before focusing on the precise definition of the disentangling algebra and the disentangling maps, we will give two examples of decomposing exponential disentanglements that have arisen in our previous work. In the first example, we use the final equality in Lemma 5.4 of [3] in the case where \( g \) is the exponential function, \( \xi_1 = \cdots = \xi_n = 1 \), and \( v = \mu_1 = \cdots = \mu_n \), to write

\[
T_{\nu, \ldots, \nu}(e^{\hat{A}_1 + \cdots + \hat{A}_n}) = e^{A_1 + \cdots + A_n},
\]

that is, when the continuous probability measures are all the same, the disentangling gives the usual exponential of the sum of operators. Thus for \( 0 < t < 1 \), we have

\[
T_{\nu, \ldots, \nu}(e^{\hat{A}_1 + \cdots + \hat{A}_n}) = e^{A_1 + \cdots + A_n} = e^{(A_1 + \cdots + A_n)\nu([0, t])}e^{(A_1 + \cdots + A_n)\nu([0, 1])} = T_{(0, v)}(e^{\hat{A}_1 + \cdots + \hat{A}_n})T_{(v, \ldots, v)}(e^{\hat{A}_1 + \cdots + \hat{A}_n})
\]

where \( v_1 = [0, t] \) and \( v_2 = [t, 1] \). The general result of this type has a considerably more complicated proof; the result is given as Corollary 3.5 but rests on Theorem 3.4.

Now we consider the second illustration that comes out of earlier work. Let \( \mu \) and \( \nu \) be continuous probability measures on \( B([0, 1]) \) admitting the decompositions \( \mu = \sum_{j=0}^{k} \mu_j \), \( v = \sum_{j=1}^{k} v_j \) with \( \mu_j \) supported within \( [a_j, b_j] \) for all \( j = 0, \ldots, k \) and \( v_j \) supported within \( [b_{j-1}, a_j] \) for all \( j = 1, \ldots, k \), where \( 0 = a_0 < b_0 < a_1 < b_1 < \cdots < a_k < b_k = 1 \). Then according to (2.7) in [4, Example 2.2], we have

\[
T_{\mu, v}e^{(\hat{A} + \hat{B})} = e^{\|\mu\|A}e^{\|v\|B}e^{\|\mu_{k-1}\|A} \cdots e^{\|v_1\|B}e^{\|\mu_0\|A},
\]

where \( \|\mu_j\| = \mu((a_j, b_j)) \), \( j = 0, \ldots, k \) and \( \|v_j\| = v([b_{j-1}, a_j]), \ j = 1, \ldots, k \). (The formula (1.2) rests on Proposition 2.2 of [4] whose proof is only sketched there; indeed, the proof is not entirely transparent. However, once we have Corollary 3.5 of this paper, the proof of such results is simple; see Example 4.5 below.)

As a consequence of Theorem 3.4, our main result, we will see that

\[
T_{\mu_1, \ldots, \mu_n}(e^{\hat{A}_1 + \cdots + \hat{A}_n}) = T_{\mu_1, 2, \ldots, \mu_n}(e^{\hat{A}_1 + \cdots + \hat{A}_n})T_{\mu_1, 1, \ldots, \mu_n, 1}(e^{\hat{A}_1 + \cdots + \hat{A}_n})
\]

for \( \mu_{j, 1} = [0, t], \mu_j \) and \( \mu_{j, 2} = [t, 1], \mu_j \), \( j = 1, \ldots, n \). According to formula (1.3) the operators

\[
K_{s, t} = T_{[s, t] \mu_1, \ldots, [s, t] \mu_n}(e^{\hat{A}_1 + \cdots + \hat{A}_n}), \quad 0 \leq s < t \leq 1,
\]

form a time-dependent family of propagators.

Note that (1.3) allows \( n \) operator-measure pairs but involves only two subintervals of \([0, 1]\). Theorem 3.4 and Corollary 3.5 permit any finite number \( n \) of operator-measure pairs and also any finite number \( d \) of subintervals.
A brief sketch of basic definitions and some elementary facts from the approach to Feynman’s operational calculi initiated in [2, 3] follows.

Let $X$ be a Banach space and let $A_1, \ldots, A_n$ be nonzero bounded linear operators on $X$. Except for the numbers $\|A_1\|, \ldots, \|A_n\|$, which will serve as weights, we ignore for the present the nature of $A_1, \ldots, A_n$ as operators and introduce a commutative Banach algebra consisting of ‘analytic functions’ $f(\tilde{A}_1, \ldots, \tilde{A}_n)$, where $\tilde{A}_1, \ldots, \tilde{A}_n$ are treated as purely formal commuting objects.

Consider the collection $D = D(A_1, \ldots, A_n)$ of all expressions of the form

$$f(\tilde{A}_1, \ldots, \tilde{A}_n) = \sum_{m_1, \ldots, m_n=0}^{\infty} c_{m_1, \ldots, m_n} \tilde{A}_1^{m_1} \cdots \tilde{A}_n^{m_n}$$

(1.4)

where $c_{m_1, \ldots, m_n} \in \mathbb{C}$ for all $m_1, \ldots, m_n = 0, 1, \ldots$, and

$$\|f(\tilde{A}_1, \ldots, \tilde{A}_n)\| = \|f(\tilde{A}_1, \ldots, \tilde{A}_n)\|_{D(A_1, \ldots, A_n)} := \sum_{m_1, \ldots, m_n=0}^{\infty} |c_{m_1, \ldots, m_n}| \|A_1\|^{m_1} \cdots \|A_n\|^{m_n} < \infty.$$  

(1.5)

The function on $D(A_1, \ldots, A_n)$ defined by (1.4) makes $D(A_1, \ldots, A_n)$ into a commutative Banach algebra under pointwise operations. We refer to $D(A_1, \ldots, A_n)$ as the disentangling algebra associated with the $n$-tuple $(A_1, \ldots, A_n)$ of bounded linear operators acting on $X$.

Let $A_1, \ldots, A_n$ be nonzero operators from $\mathcal{L}(X)$ and let $\mu_1, \ldots, \mu_n$ be continuous measures defined at least on $\mathcal{B}([0, 1])$, the Borel class of $[0, 1]$. Usually these are taken as probability measures so that $\tilde{A}_1$, say, is disentangled to $A_1\mu_1[0, 1] = A_1$, but for the present purpose we do not insist on this. The total mass of a measure $\mu$ is written as $\|\mu\|$. The idea is to replace the operators $A_1, \ldots, A_n$ with the elements $\tilde{A}_1, \ldots, \tilde{A}_n$ from $D$ and then form the desired function of $\tilde{A}_1, \ldots, \tilde{A}_n$. Still working in the commutative algebra $D$, we time order the expression for the function and then pass back to the noncommutative algebra $\mathcal{L}(X)$ simply by removing the tildes. This set-up will permit us to work rigorously while staying in the spirit of Feynman’s heuristic ideas. (See [1, 2] and Chap. 14 of [6] for a discussion of Feynman’s heuristic ideas.)

Given nonnegative integers $m_1, \ldots, m_n$, we let $m = m_1 + \cdots + m_n$ and

$$P^{m_1, \ldots, m_n}(z_1, \ldots, z_n) = z_1^{m_1} \cdots z_n^{m_n}.$$  

(1.6)

We are now ready to formally define the disentangling map $T_{\mu_1, \ldots, \mu_n}$ which will return us from our commutative framework to the noncommutative setting of $\mathcal{L}(X)$. For $j = 1, \ldots, n$ and all $s \in [0, 1]$, we take $A_j(s) = A_j$ and, for $i = 1, \ldots, m$, we define

$$C_i(s) := \begin{cases} 
A_1(s) & \text{if } i \in \{1, \ldots, m_1\}, \\
A_2(s) & \text{if } i \in \{m_1 + 1, \ldots, m_1 + m_2\}, \\
\vdots & \\
A_n(s) & \text{if } i \in \{m_1 + \cdots + m_{n-1} + 1, \ldots, m\}.
\end{cases}$$

(1.7)
For each $m = 0, 1, \ldots$, let $S_m$ denote the set of all permutations of the integers $\{1, \ldots, m\}$, and given $\pi \in S_m$, we let

$$\Delta_m(\pi) = \{(s_1, \ldots, s_m) \in [0, 1]^m : 0 < s_{\pi(1)} < \cdots < s_{\pi(m)} < 1\}.$$

**Definition 1.1**

$$T_{\mu_1, \ldots, \mu_n} \left( P^{m_1, \ldots, m_n}(\tilde{A}_1, \ldots, \tilde{A}_n) \right) := \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)})(\mu_1^{m_1} \times \cdots \times \mu_n^{m_n})(ds_1, \ldots, ds_m).$$  \hfill (1.8)

The notation $\mu_j^0$ means that the integral with respect to the $s_j$-variable is simply omitted. We adopt this convention even if $\mu_j$ is the zero measure.

Then, for $f(\tilde{A}_1, \ldots, \tilde{A}_n) \in \mathbb{D}(A_1, \ldots, A_n)$ given by

$$f(\tilde{A}_1, \ldots, \tilde{A}_n) = \sum_{m_1, \ldots, m_n = 0}^{\infty} c_{m_1, \ldots, m_n} \tilde{A}_1^{m_1} \cdots \tilde{A}_n^{m_n},$$  \hfill (1.9)

we set $T_{\mu_1, \ldots, \mu_n}(f(\tilde{A}_1, \ldots, \tilde{A}_n))$ equal to

$$\sum_{m_1, \ldots, m_n = 0}^{\infty} c_{m_1, \ldots, m_n} \left( T_{\mu_1, \ldots, \mu_n} \left( P^{m_1, \ldots, m_n}(\tilde{A}_1, \ldots, \tilde{A}_n) \right) \right).$$  \hfill (1.10)

If $A_1, \ldots, A_n$ commute and the measures are probabilities, then the right-hand side of (1.8) equals $A_1^{m_1} \cdots A_n^{m_n}$, which is what we would expect [2, Proposition 2.2]. As is usual, we shall write the operator $T_{\mu_1, \ldots, \mu_n} f$ in place of $T_{\mu_1, \ldots, \mu_n}(f)$ for an element $f$ of $\mathbb{D}(A_1, \ldots, A_n)$.

We shall sometimes write the bounded linear operator

$$T_{\mu_1, \ldots, \mu_n}(f(\tilde{A}_1, \ldots, \tilde{A}_n))$$

as $f_{\mu_1, \ldots, \mu_n}(A_1, \ldots, A_n)$. In particular,

$$P^{m_1, \ldots, m_n}_{\mu_1, \ldots, \mu_n}(A_1, \ldots, A_n) = T_{\mu_1, \ldots, \mu_n} \left( P^{m_1, \ldots, m_n}(\tilde{A}_1, \ldots, \tilde{A}_n) \right).$$  \hfill (1.11)

Now take $n = 2$ in formula (1.3). From (1.10) we can write

$$T_{\mu_1, \mu_2}(e^{\tilde{A}_1 + \tilde{A}_2}) = \sum_{m_1, m_2 = 0}^{\infty} \frac{1}{m_1! m_2!} p^{m_1, m_2}_{\mu_1, \mu_2}(A_1, A_2)$$

with respect to the disentangling $p^{m_1, m_2}_{\mu_1, \mu_2}(A_1, A_2)$ defined by (1.11) for $m_1, m_2 = 0, 1, \ldots$.

To prove formula (1.3) in the case $n = 2$, it suffices to establish that

$$p^{m_1, m_2}_{\mu_1, \mu_2}(A_1, A_2) = \sum_{i_1 + j_1 = m_1, i_2 + j_2 = m_2} \frac{m_1! m_2!}{i_1! j_1! i_2! j_2!} p^{j_1, j_2}_{\mu_1,2,2}(A_1, A_2) p^{i_1, i_2}_{\mu_1,1,2,1}(A_1, A_2),$$  \hfill (1.12)

on expanding the exponential $e^{\tilde{A}_1 + \tilde{A}_2}$ in a multiple Taylor series about zero in the disentangling algebra. The sum is over nonnegative integers $i_k, j_k$ for $k = 1, 2$. 

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Disentangling is the central operation of Feynman’s operational calculus and, as Feynman notes, is often difficult to carry out. How might one discover the formula in Theorem 3.4, our main result? It (along with formulas in various earlier papers) was discovered by using heuristic ideas from [1]. The ideas were discussed briefly in [2] and in more detail in Chap. 14 of [6]. We now show how to ‘derive’ the formula for the case \( n = 2 \) of Theorem 3.4 via the heuristics. The actual proof requires going back to Definition 1.1 or, when possible, using previous results.

Our goal now is to ‘derive’ the formula (1.12). We consider the elements \( \tilde{A}_1 \) and \( \tilde{A}_2 \) in the commutative disentangling algebra \( D(A_1, A_2) \) and we attach time indices to \( \tilde{A}_1 \) and \( \tilde{A}_2 \) as indicated in the left-hand side of (1.12):

\[
\tilde{A}_1 = \int_0^1 \tilde{A}_1(s) \mu_1(ds), \quad \tilde{A}_2 = \int_0^1 \tilde{A}_2(s) \mu_2(ds).
\]

Now we calculate within \( D(A_1, A_2) \) and rearrange until the time-ordering coincides with the ordering on the page, so that

\[
P_{m_1, m_2}(\tilde{A}_1, \tilde{A}_2) = \left( \int_0^1 \tilde{A}_1(s) \mu_1(ds) \right)^{m_1} \left( \int_0^1 \tilde{A}_2(s) \mu_2(ds) \right)^{m_2}
\]

\[
= \left( \int_0^a \tilde{A}_1(s) \mu_1(ds) + \int_a^1 \tilde{A}_1(s) \mu_1(ds) \right)^{m_1}
\]

\[
\times \left( \int_0^a \tilde{A}_2(s) \mu_2(ds) + \int_a^1 \tilde{A}_2(s) \mu_2(ds) \right)^{m_2}
\]

\[
= \sum_{i_1+j_1=m_1 \atop i_2+j_2=m_2} \frac{m_1! \cdot m_2!}{i_1! \cdot j_1! \cdot i_2! \cdot j_2!} \left( \int_a^1 \tilde{A}_1(s) \mu_1(ds) \right)^{j_1} \left( \int_a^1 \tilde{A}_2(s) \mu_2(ds) \right)^{j_2}
\]

\[
\times \left( \int_0^a \tilde{A}_1(s) \mu_1(ds) \right)^{i_1} \left( \int_0^a \tilde{A}_2(s) \mu_2(ds) \right)^{i_2}.
\]

Now the two products

\[
\left( \int_a^1 \tilde{A}_1(s) \mu_1(ds) \right)^{j_1} \left( \int_a^1 \tilde{A}_2(s) \mu_2(ds) \right)^{j_2}
\]

and

\[
\left( \int_0^a \tilde{A}_1(s) \mu_1(ds) \right)^{i_1} \left( \int_0^a \tilde{A}_2(s) \mu_2(ds) \right)^{i_2}
\]

are disentangled by definition to \( P_{j_1, j_2}^{i_1, i_2}(A_1, A_2) \) and \( P_{i_1, i_2}^{j_1, j_2}(A_1, A_2) \), respectively. Thus (1.12) has been ‘derived’ heuristically. The formal proof of Theorem 3.4 below is largely based on the same ideas. The key is a somewhat technical combinatorial result (Lemma 3.3) whose formulation is inspired by (1.12).

2 A Rigorous Proof in the Case of 2 Operators

In the case \( n = 2 \) we can give a simple direct proof of formula (1.3) by a perturbation series [3, Corollary 5.2].
Theorem 2.1 Let $X$ be a Banach space and let $\mu$ and $\nu$ be continuous measures on the Borel $\sigma$-algebra $\mathcal{B}[0,1]$ of $[0,1]$. Let $A, B$ be elements of $\mathcal{L}(X)$.

Fix $t \in (0,1)$ and let $\mu = \mu_1 + \mu_2$, $\nu = \nu_1 + \nu_2$ be the decompositions of $\mu$, $\nu$ with respect to $[0, t]$ and its complement, that is, $\mu_1 = [0, t], \mu, \nu_1 = [0, t], \nu, \mu_2 = [t, 1], \mu, \nu_2 = [t, 1], \nu$. Then

$$T_{\mu, \nu}(e^{A+B}) = T_{\mu_2, \nu_2}(e^{A+B})T_{\mu_1, \nu_1}(e^{A+B}).$$  \hspace{1cm} (2.1)

Proof For $k = 1, 2, \ldots$, let $\Delta_k = \{(s_1, \ldots, s_k) \in [0, 1]^k : 0 < s_1 < \cdots < s_k < 1\}$ and $\Delta_0 = [0, 1]$. The sets

$$\Delta_k(m; t) = \{(s_1, \ldots, s_k) \in [0, 1]^k : 0 < s_1 < \cdots < s_m < t < s_{m+1} < \cdots < s_k < 1\},$$

for $m = 0, \ldots, k$, are pairwise disjoint and up to a set of $\nu^k$-measure zero, we have

$$\Delta_k = \bigcup_{m=0}^{k} \Delta_k(m; t).$$

Clearly $\Delta_k(m; t) \subset \Delta_k$ and are pairwise disjoint for $m = 0, \ldots, k$. On the other hand, suppose that $\xi \in \Delta_k$ and no coordinate of $\xi$ equals $t$. Then $t \in \bigcup_{m=0}^{k} (\xi_m, \xi_{m+1})$ with $\xi_0 = 0$ and $\xi_{k+1} = 1$. Consequently $\xi \in \Delta_k(j; t)$ for the unique $j = 0, \ldots, k$ such that $t \in (\xi_j, \xi_{j+1})$ and so $\xi$ is an element of $\bigcup_{m=0}^{k} \Delta_k(m; t)$.

If $m \neq 0, k$, then $\Delta_k(m; t)$ is the product set

$$\{0 < s_1 < \cdots < s_m < t\} \times \{t < s_{m+1} < \cdots < s_k < 1\}.$$  \hspace{1cm} (2.2)

According to [3, Corollary 5.2] and the paragraph following it, we have

$$T_{\mu, \nu}(e^{A+B}) = \sum_{k=0}^{\infty} \left[ \int_{\Delta_k} e^{A_{\mu}(\{s_1, 1\})} B e^{A_{\mu}(\{s_1-1, s_k\})} \cdots B e^{A_{\mu}(\{0, s_1\})} \nu^k(d s_1, \ldots, d s_k) \right].$$

the first term being $e^{1/\mu A}$. With the understanding that a factor involving an “integral” with respect to the power $\mu^0$ of a measure $\mu$ is omitted, by Fubini’s theorem we have

$$\int_{\Delta_k} e^{A_{\mu}(\{s_1, 1\})} B e^{A_{\mu}(\{s_1-1, s_k\})} \cdots B e^{A_{\mu}(\{0, s_1\})} \nu^k(d s_1, \ldots, d s_k)$$

$$= \sum_{m=0}^{k} \int_{\Delta_k(m; t)} e^{A_{\mu}(\{s_1, 1\})} B e^{A_{\mu}(\{s_1-1, s_k\})} \cdots B e^{A_{\mu}(\{0, s_1\})} \nu^k(d s_1, \ldots, d s_k)$$

$$= \sum_{m+n=k} \int_{\{t < s_{m+1} < \cdots < s_k < 1\}} e^{A_{\mu}(\{s_1, 1\})} B e^{A_{\mu}(\{s_1-1, s_k\})} \cdots B e^{A_{\mu}(\{t, s_m+1\})}$$

$$\times \left( \int_{\{0 < s_1 < \cdots < s_m < t\}} e^{A_{\mu}(\{s_m, 1\})} B e^{A_{\mu}(\{s_{m-1}, s_m\})} \right) \nu_{1}^{n}(d s_{m+1}, \ldots, d s_k),$$

because $e^{A_{\mu}(\{s_m, s_{m+1}\})} = e^{A_{\mu}(\{t, s_{m+1}\})} e^{A_{\mu}(\{s_m, t\})}$ for $s_m < t < s_{m+1}$ in each integral.
It follows that

\[ T_{\mu, \nu} \left( e^{\hat{A} + \hat{B}} \right) = \sum_{k=0}^{\infty} \sum_{m+n=k} \int_{\Delta_k[0,1]} e^{A_{\mu_2}(\{s_m, 1\})} B e^{A_{\mu_2}(\{s_{m-1}, s_m\})} \ldots B e^{A_{\mu_2}(\{s_1, 1\})} v^m_{\mu_2}(ds_1, \ldots, ds_n) \times \int_{\Delta_m[0,1]} e^{A_{\mu_1}(\{s_m, 1\})} B e^{A_{\mu_1}(\{s_{m-1}, s_m\})} \ldots B e^{A_{\mu_1}(\{s_1, 1\})} v^m_{\mu_1}(ds_1, \ldots, ds_n) \]

\[ = \sum_{m,n=0}^{\infty} \int_{\Delta_m[0,1]} e^{A_{\mu_2}(\{s_m, 1\})} B e^{A_{\mu_2}(\{s_{m-1}, s_m\})} \ldots B e^{A_{\mu_2}(\{s_1, 1\})} v^m_{\mu_2}(ds_1, \ldots, ds_n) \times \int_{\Delta_m[0,1]} e^{A_{\mu_1}(\{s_m, 1\})} B e^{A_{\mu_1}(\{s_{m-1}, s_m\})} \ldots B e^{A_{\mu_1}(\{s_1, 1\})} v^m_{\mu_1}(ds_1, \ldots, ds_n) \]

\[ = \left( \sum_{n=0}^{\infty} \int_{\Delta_m[0,1]} e^{A_{\mu_2}(\{s_m, 1\})} B e^{A_{\mu_2}(\{s_{m-1}, s_m\})} \ldots B e^{A_{\mu_2}(\{s_1, 1\})} v^m_{\mu_2}(ds_1, \ldots, ds_n) \right) \times \left( \sum_{m=0}^{\infty} \int_{\Delta_m[0,1]} e^{A_{\mu_1}(\{s_m, 1\})} B e^{A_{\mu_1}(\{s_{m-1}, s_m\})} \ldots B e^{A_{\mu_1}(\{s_1, 1\})} v^m_{\mu_1}(ds_1, \ldots, ds_n) \right) \]

\[ = T_{\mu_2, v_2} \left( e^{\hat{A} + \hat{B}} \right) T_{\mu_1, v_1} \left( e^{\hat{A} + \hat{B}} \right). \]

\[ \square \]

**Corollary 2.2** Let X be a Banach space and let \( \mu \) and \( \nu \) be continuous measures on the Borel \( \sigma \)-algebra \( \mathcal{B}[0,1] \) of \([0,1]\). Let \( A, B \) be elements of \( \mathcal{L}(X) \).

Fix \( t \in (0, 1) \) and let \( \mu = \mu_1 + \mu_2, \nu = \nu_1 + \nu_2 \) be the decompositions of \( \mu, \nu \) with respect to \([0, t]\) and its complement, that is, \( \mu_1 = [0, t], \mu, \nu_1 = [0, t], \nu \), \( \mu_2 = [t, 1], \mu, \nu_2 = [t, 1], \nu \). Then for all nonnegative integers \( m, n \), we have

\[ P^m_{\mu, v}(A, B) = \sum_{i_1 + i_2 = m} \sum_{j_1 + j_2 = n} \frac{m!}{i_1! j_1! i_2! j_2!} P^j_{\mu_2, v_2}(A, B) P^{i_1, i_2}_{\mu_1, v_1}(A, B). \]  

(2.3)

**Proof** Let \( z_1, z_2 \in \mathbb{C} \). By virtue of formula (1.10) we have

\[ T_{\mu_2, v_2} \left( e^{(z_1 \hat{A} + z_2 \hat{B})} \right) T_{\mu_1, v_1} \left( e^{(z_1 \hat{A} + z_2 \hat{B})} \right) \]

\[ = \left( \sum_{i_1, j_2 = 0}^{\infty} \frac{z_1^{i_1} z_2^{j_2}}{i_1! j_1!} P^j_{\mu_2, v_2}(A, B) \right) \left( \sum_{i_1, j_2 = 0}^{\infty} \frac{z_1^{i_1} z_2^{j_2}}{i_1! j_1!} P^j_{\mu_2, v_2}(A, B) \right) \]

\[ = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{i_1 + i_2 = m} \sum_{j_1 + j_2 = n} \frac{m!}{i_1! j_1! i_2! j_2!} P^j_{\mu_2, v_2}(A, B) P^{i_1, i_2}_{\mu_1, v_1}(A, B) \]

and

\[ T_{\mu, \nu} \left( e^{(z_1 \hat{A} + z_2 \hat{B})} \right) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{i_1 + i_2 = m} \sum_{j_1 + j_2 = n} \frac{m! n!}{i_1! j_1! i_2! j_2!} P^m_{\mu, v}(A, B). \]
According to Theorem 2.1, we may equate coefficients of the double power series. This proves formula (2.3).

Combined with induction, the argument of Theorem 2.1 would work for more than two operators provided that we had the following generalization of Corollary 5.2 from [3].

If λ is a measure and A is a λ-measurable set, we denote the restricted measure $B \mapsto \lambda(A \cap B)$ by $A_\lambda$.

**Theorem 2.3** Let $X$ be a Banach space. Fix a positive integer $n$ and let $\mu_1, \ldots, \mu_n$ be finite continuous measures on the Borel σ-algebra $B[0,1]$ of $[0,1]$. Let $A_1, \ldots, A_n, B$ be elements of $\mathcal{L}(X)$. Then

$$
T_{\mu_1, \ldots, \mu_n, \nu}(e^{\tilde{A}_1 + \cdots + \tilde{A}_n + \tilde{B}}) = \sum_{k=0}^{\infty} \int_{\Delta_k} T_{[s_1,1], \mu_1, \ldots, [s_k,1], \mu_n}(e^{\tilde{A}_1 + \cdots + \tilde{A}_n}) B T_{[s_{k-1},s_k], \mu_1, \ldots, [s_{k-1},s_k], \mu_n}(e^{\tilde{A}_1 + \cdots + \tilde{A}_n}) \cdots
$$

$$
\cdots T_{[s_{1,2}], \mu_1, \ldots, [s_{1,2}], \mu_n}(e^{\tilde{A}_1 + \cdots + \tilde{A}_n}) B T_{[0,s_1], \mu_1, \ldots, [0,s_1], \mu_n}(e^{\tilde{A}_1 + \cdots + \tilde{A}_n}) v^k(ds_1, \ldots, ds_k).
$$

(2.4)

For $n = 1$, we have $T_{I, \mu_1}(e^{\tilde{A}_1}) = e^{\mu_1(I)A_1}$ for all subintervals $I$ of $[0,1]$ and Theorem 2.3 for $n = 1$ is proved in [3, Corollary 5.2]. It seems more direct to prove Theorem 3.4 below from the disentangling formula Definition 1.1; we shall derive Theorem 2.3 from an associated result, Theorem 3.6.

### 3 Splitting Polynomials in $n$ Operators

An essential part of the proof of Theorem 2.1 is the decomposition of $\Delta_k$ into $(k + 1)$ disjoint product sets $\Delta_k(m; t)$, $m = 0, 1, \ldots, k$, and the application of Fubini’s Theorem with respect to the product measure $\nu^k = \nu \times \cdots \times \nu$, for each $k = 2, 3, \ldots$. Now fix $t \in (0, 1)$. We first consider the case $n = 2$ in formula (1.8) with two continuous measures $\mu$ and $\nu$. Let $\mu_1 = \mu_1 + \mu_2$, $\nu = v_1 + v_2$ be the decompositions of $\mu, \nu$ with respect to $[0, t]$ and its complement, that is, $\mu_1 = [0, t], \mu, v_1 = [0, t], v, \mu_2 = [t, 1], \mu, v_2 = [t, 1], v$. Given nonnegative integers $m_1$ and $m_2$ such that $m = m_1 + m_2$ and a permutation $\pi$ in $S_m$, up to a set of $(\mu^{m_1} \times \nu^{m_2})$-measure zero, the subset

$$
\Delta_m(\pi) = \{(s_1, \ldots, s_m) \in [0,1]^m : 0 < s_{\pi(1)} < \cdots < s_{\pi(m)} < 1\}
$$

of $[0,1]^m$ can be decomposed into the disjoint union of sets $\Delta_m(\pi; i_1, i_2; j_1, j_2)$ of elements $(s_1, \ldots, s_m) \in \Delta_m(\pi)$ such that exactly $i_1$ of the times $s_1, \ldots, s_{m_1}$ belong to $[0, t)$ and $i_2$ of the times $s_{m_1+1}, \ldots, s_m$ belong to $[0, t)$ and no time $s_j$ equals $t$ for $j = 1, \ldots, m$. In this notation, $j_1 = m_1 - i_1$ and $j_2 = m_2 - i_2$ for $i_1 = 0, \ldots, m_1$ and $i_2 = 0, \ldots, m_2$. Then for each $(s_1, \ldots, s_m) \in \Delta_m(\pi; i_1, i_2; j_1, j_2)$, there are exactly $i_1 + i_2$ elements of the set $\{s_1, \ldots, s_m\}$ less than $t$, so we must have

$$
0 < s_{\pi(1)} < \cdots < s_{\pi(i_1+i_2)} < t < s_{\pi(i_1+i_2+1)} < \cdots < s_{\pi(m)}.
$$
Let
\[ U_1 = \{ \pi(1), \ldots, \pi(i_1 + i_2) \} \cap \{ 1, \ldots, m_1 \}, \quad V_1 = \{ 1, \ldots, m_1 \} \setminus U_1, \]
\[ U_2 = \{ \pi(1), \ldots, \pi(i_1 + i_2) \} \cap \{ m_1 + 1, \ldots, m \}, \quad V_2 = \{ m_1 + 1, \ldots, m \} \setminus U_2. \]

The set \( \Delta_m(\pi; i_1, i_2; j_1, j_2) \) is nonempty if and only if \( U_1 \) has \( i_1 \) elements and \( U_2 \) has \( i_2 \) elements. Up to a permutation of coordinates, the subset \( \Delta_m(\pi; i_1, i_2; j_1, j_2) \) of \( [0, 1]^m \) is either empty or if \( i_1 + i_2 \neq 0, m \), it can be identified up to a set of measure zero with the Cartesian product of the two sets
\[ \{ ((s_j)_{j \in U_1}, (s_j)_{j \in U_2}) : 0 < s_{\pi(1)} < \cdots < s_{\pi(i_1+i_2)} < t \}, \]
\[ \{ ((s_k)_{k \in V_1}, (s_k)_{k \in V_2}) : t < s_{\pi(i_1+i_2+1)} < \cdots < s_{\pi(m)} < 1 \} \]
and the restriction of \( \mu^{m_1} \times \mu^{m_2} \) to \( \Delta_m(\pi; i_1, i_2; j_1, j_2) \) is the product measure
\[ \left( (\mu_1^{i_1} \times v^{j_1})(ds_j)_{j \in U_1}, (ds_j)_{j \in U_2} \right) \times \left( (\mu_2^{i_2} \times v^{j_2})(ds_j)_{j \in V_1}, (ds_j)_{j \in V_2} \right). \]

In the case that \( \Delta_m(\pi; i_1, i_2; j_1, j_2) \) is nonempty, an appeal to Fubini’s theorem gives a decomposition of formula (1.8) into the operator product of an integral with respect to the measure \( \mu_1^{i_1} \times v^{j_1} \) followed by an integral with respect to the measure \( \mu_2^{i_2} \times v^{j_2} \).

The equality (2.3) is obtained once we identify the sets \( \Delta_m(\pi; i_1, i_2; j_1, j_2) \) which are nonempty for given \( \pi \in \mathcal{S}_m \). We first introduce some notation for the general case of \( n \) operators and measures in order to decompose the integrals appearing in Definition 1.1.

Let \( 0 < t < 1 \) and let \( m_1, \ldots, m_n \) be nonnegative integers with \( m = m_1 + \cdots + m_n \) and \( m_0 = 0 \). For each \( i_k = 0, \ldots, m_k \) and \( j_k = m_k - i_k \) defined for \( k = 1, \ldots, n \), the sets
\[ \Delta_m(\pi; i_1, \ldots, i_n; j_1, \ldots, j_n) \]
\[ = \{(s_1, \ldots, s_m) \in [0, 1]^m : i_1 \text{ of } s_1, \ldots, s_{m_1} \text{ and } i_j \text{ of } s_{m_1+\cdots+m_{j-1}+1}, \ldots, s_{m_1+\cdots+m_j} \text{ belong to } (0, t) \text{ for } j = 2, \ldots, n \text{ and } 0 < s_{\pi(1)} < \cdots < s_{\pi(i_1+i_2+i_3+\cdots+i_n)} < t < s_{\pi(i_1+i_2+i_3+\cdots+i_n+1)} < \cdots < s_{\pi(m)} < 1 \} \]
are contained in \( \Delta_m(\pi) \). In fact we have

**Lemma 3.1** Up to a \( \mu_1^{m_1} \times \cdots \times \mu_n^{m_n} \)-null set, \( \Delta_m(\pi) \) is the union of the pairwise disjoint sets \( \Delta_m(\pi; i_1, \ldots, i_n; j_1, \ldots, j_n) \), for some nonnegative integers \( i_k + j_k = m_k \) and \( k = 1, \ldots, n \).

**Proof** Clearly \( \Delta_m(\pi) \) contains the union. Now suppose that \( (s_1, \ldots, s_m) \in \Delta_m(\pi) \) and no coordinate equals 0, 1 or \( t \). Let \( i_1 = 0, \ldots, m \) be the cardinality of \( \{ s_1, \ldots, s_m \} \cap (0, t) \) and let \( i_j = 0, \ldots, n \) be the cardinality of \( \{ s_{m_1+\cdots+m_{j-1}+1}, \ldots, s_{m_1+\cdots+m_j} \} \cap (0, t) \). Then exactly \( i_1 + \cdots + i_n \) elements of \( \{ s_1, \ldots, s_m \} \) belong to the interval \((0, t)\). Because \( (s_1, \ldots, s_m) \in \Delta_m(\pi) \), these elements must be the set of \((i_1 + \cdots + i_n)\) numbers \( s_{\pi(1)}, \ldots, s_{\pi(i_1+i_2+i_3+\cdots+i_n)} \), so that \( (s_1, \ldots, s_m) \in \Delta_m(\pi; i_1, \ldots, i_n; j_1, \ldots, j_n) \). Because \( \mu_1, \ldots, \mu_n \) are assumed to be continuous measures, the set of all such \((s_1, \ldots, s_m) \in \Delta_m(\pi)\) has full \( \mu_1^{m_1} \times \cdots \times \mu_n^{m_n} \)-measure [2, Lemma 2.1]. \( \square \)

Suppose that \( \Delta_m(\pi; i_1, \ldots, i_n; j_1, \ldots, j_n) \) is nonempty and \( (s_1, \ldots, s_m) \) is an element of \( \Delta_m(\pi; i_1, \ldots, i_n; j_1, \ldots, j_n) \). Then there are exactly \( i_1 + \cdots + i_n \) elements of the set
\{s_1, \ldots, s_m\} \text{ less than } t, \text{ namely } 0 < s_{\pi(1)} < \cdots < s_{\pi(i_1 + \cdots + i_n)} < t \text{ where }

\begin{align}
  i_1 &= \#(\{\pi(1), \ldots, \pi(i_1 + \cdots + i_n)\} \cap \{1, \ldots, m_1\}) \\
  \vdots \\
  i_n &= \#(\{\pi(1), \ldots, \pi(i_1 + \cdots + i_n)\} \cap \{m_1 + \cdots + m_{n-1} + 1, \ldots, m\}).
\end{align}  

(3.1)

Let \(S_m(i_1, \ldots, i_n; j_1, \ldots, j_n)\) denote the set of permutations \(\pi \in S_m\) for which (3.1) hold, that is,

\[S_m(i_1, \ldots, i_n; j_1, \ldots, j_n) = \{\pi \in S_m : \Delta_m(\pi; i_1, \ldots, i_n; j_1, \ldots, j_n) \neq \emptyset\}.\]  

(3.2)

For each \(\pi \in S_m(i_1, \ldots, i_n; j_1, \ldots, j_n)\), the nonempty open set \(\Delta_m(\pi; i_1, \ldots, i_n; j_1, \ldots, j_n)\) is equal to

\[\{(s_1, \ldots, s_m) : 0 < s_{\pi(1)} < \cdots < s_{\pi(i_1 + \cdots + i_n)} < t < s_{\pi(i_1 + \cdots + i_n + 1)} < \cdots < s_{\pi(m)} < 1\}.\]

Let \(\pi \in S_m(i_1, \ldots, i_n; j_1, \ldots, j_n)\) and define the sets \(U_1, \ldots, U_n\) by

\[U_1 = \{\pi(1), \ldots, \pi(i_1 + \cdots + i_n)\} \cap \{1, \ldots, m_1\} \]

\[\vdots \]

\[U_n = \{\pi(1), \ldots, \pi(i_1 + \cdots + i_n)\} \cap \{m_1 + \cdots + m_{n-1} + 1, \ldots, m\}.
\]

(3.3)

The sets \(V_1, \ldots, V_n\) are defined by \(V_1 = \{1, \ldots, m_1\} \setminus U_1\) and

\[V_j = \{m_1 + \cdots + m_{j-1} + 1, \ldots, m_1 + \cdots + m_j\} \setminus U_j, \quad j = 2, \ldots, n.\]  

(3.4)

Up to a permutation of coordinates, the measure \((\mu_1^{m_1} \times \cdots \times \mu_n^{m_n})(ds_1, \ldots, ds_m)\) is equal to the product measure

\[\mu_1^{i_1} \times \cdots \times \mu_1^{i_n}((ds_j)_{j \in U_1}, \ldots, (ds_j)_{j \in U_n}) \]

\[\times (\mu_1^{j_1} \times \cdots \times \mu_1^{j_n})((ds_j)_{j \in V_1}, \ldots, (ds_j)_{j \in V_n})).\]

The proof of the following observation follows immediately from the definitions.

**Lemma 3.2** Let \(\pi \in S_m(i_1, \ldots, i_n; j_1, \ldots, j_n)\) and suppose that the sets \(U_j, V_j, j = 1, \ldots, n\) are defined by (3.3) and (3.4). Let the sets \(U_1 \cup \cdots \cup U_n\) and \(V_1 \cup \cdots \cup V_n\) be enumerated as increasing integers in the case that they are nonempty.

The set \(\Delta_m(\pi; i_1, \ldots, i_n; j_1, \ldots, j_n)\) is equal to the cartesian product of the two sets

\[\{(s_j)_{j \in U_1}, \ldots, (s_j)_{j \in U_n} : 0 < s_{\pi(1)} < \cdots < s_{\pi(i_1 + \cdots + i_n)} < t \};\]  

(3.5)

\[\{(s_k)_{k \in V_1}, \ldots, (s_k)_{k \in V_n} : t < s_{\pi(i_1 + \cdots + i_n + 1)} < \cdots < s_{\pi(m)} < 1\} \]

(3.6)

followed by the permutation of coordinates sending the coordinate vector

\[(s_j)_{j \in U_1}, \ldots, (s_j)_{j \in U_n}, (s_k)_{k \in V_1}, \ldots, (s_k)_{k \in V_n}\]

to \((s_1, \ldots, s_m)\), or, if one of (3.5) or (3.6) is empty, \(\Delta_m(\pi; i_1, \ldots, i_n; j_1, \ldots, j_n)\) is equal to the other.
Suppose that $\pi \in S_m(i_1, \ldots, i_n; j_1, \ldots, j_n)$ so that the set $\Delta_m(\pi; i_1, \ldots, i_n; j_1, \ldots, j_n)$ is nonempty and that $0 < i_1 + \cdots + i_n < m$. Then by equations (3.1), neither $U = U_1 \cup \cdots \cup U_n$ nor $V = V_1 \cup \cdots \cup V_n$ is empty. First we examine formula (1.8) but with $\Delta_m(\pi)$ replaced by the set $\Delta_m(\pi; i_1, \ldots, i_n; j_1, \ldots, j_n)$.

Let $\mu_{j,1} = [0, t]$ and $\mu_{j,2} = [t, 1]$ for $j = 1, \ldots, n$ be the decompositions of $\mu_1, \ldots, \mu_n$ with respect to $[0, t]$ and its complement. An application of Fubini’s Theorem and Lemma 3.2 shows that

$$\int_{\Delta_m(\pi; i_1, \ldots, i_n; j_1, \ldots, j_n)} C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)})(\mu_1^{m_1} \times \cdots \times \mu_n^{m_n})(ds_1, \ldots, ds_m)$$

$$= \int_{[t < s_{\pi(i_1 + \cdots + i_n + 1)} < \cdots < s_{\pi(m)} < t]} (\mu_1^{j_1} \times \cdots \times \mu_n^{j_n})(ds_j \in V_1, \ldots, ds_j \in V_n))$$

$$\times \left( \int_{[0 < s_{\pi(1)} < \cdots < s_{\pi(i_1 + \cdots + i_n)} < t]} C_{\pi(1)}(\mu_1^{i_1} \times \cdots \times \mu_n^{i_n}) \right)$$

$$\times (ds_j \in U_1, \ldots, ds_j \in U_n)$$

$$= \left( \int_{[t < s_{\pi(i_1 + \cdots + i_n + 1)} < \cdots < s_{\pi(m)} < t]} C_{\pi(m)} \cdots C_{\pi(i_1 + \cdots + i_n + 1)} \right)$$

$$\times \left( \int_{[0 < s_{\pi(1)} < \cdots < s_{\pi(i_1 + \cdots + i_n)} < t]} C_{\pi(i_1 + \cdots + i_n)} \cdots C_{\pi(1)} \right)$$

$$\times (ds_j \in U_1, \ldots, ds_j \in U_n).$$

Here we have used the fact that $C_j(s_j)$ is actually independent of the parameter $s_j$, so we have dropped the parameter $s_j$ from the notation. The integrals are defined independently of any particular enumeration of $U_j$ and $V_j$.

Let $k_1 < \cdots < k_{i_1 + \cdots + i_n}$ be the increasing enumeration of $U = U_1 \cup \cdots \cup U_n$ and $\ell_1 < \cdots < \ell_{j_1 + \cdots + j_n}$ be the corresponding enumeration of $V = V_1 \cup \cdots \cup V_n$. Let $\alpha_U(k_p) = p$ for $p = 1, \ldots, i_1 + \cdots + i_n$ and $\beta_V(\ell_q) = q$ for $q = 1, \ldots, j_1 + \cdots + j_n$.

A change of variables gives

$$\int_{\Delta_m(\pi; i_1, \ldots, i_n; j_1, \ldots, j_n)} C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)})(\mu_1^{m_1} \times \cdots \times \mu_n^{m_n})(ds_1, \ldots, ds_m)$$

$$= \left( \int_{[t < s_{\beta_V \circ \pi(i_1 + \cdots + i_n + 1)} < \cdots < s_{\beta_V \circ \pi(m)} < t]} C_{\pi(m)} \cdots C_{\pi(i_1 + \cdots + i_n + 1)} \right)$$

$$\times (\mu_1^{j_1} \times \cdots \times \mu_n^{j_n})(ds_j, \ldots, ds_{j_1 + \cdots + j_n})$$

$$\times \left( \int_{[0 < s_{\alpha_U \circ \pi(1)} < \cdots < s_{\alpha_U \circ \pi(i_1 + \cdots + i_n)} < t]} C_{\pi(i_1 + \cdots + i_n)} \cdots C_{\pi(1)} \right).$$
is in one-to-one correspondence with the set
\[ Sm(U) \]
\[ \text{is a one-to-one correspondence between } V \]
\[ \text{are given by } (3.3) \]
\[ \text{via the mapping } \]
\[ \text{Let } \sigma_i \text{ denote the collection of all subsets } \]V{i_1},...,i_n \text{ of permutations } \pi, \pi \] of permutations \pi \] ∈ Sm(i \] 1, ..., i_n \] in \] to one-one correspondence with the set
\[ U(i_1, ..., i_n; j_1, ..., j_n) \]
\[ \times S_{i_1 + ... + i_n} \times S_{j_1 + ... + j_n} \]
\[ j : \pi \mapsto (U_\pi, \alpha_U \circ \pi, \beta_V \circ \pi \circ \sigma_{i_1 + ... + i_n}), \quad \pi \in Sm(i_1, ..., i_n; j_1, ..., j_n). \] (3.9)

For each \pi \] ∈ Sm(i \] 1, ..., i_n; j_1, ..., j_n), the sets U_\pi = U_1 \cup ... \cup U_n and V_\pi = V_1 \cup ... \cup V_n are given by (3.3) and (3.4).

The set \]Sm(i \] 1, ..., i_n; j_1, ..., j_n) \] is the disjoint union of the sets
\[ Sm(U), \quad U \in \mathcal{U}(i_1, ..., i_n; j_1, ..., j_n) \]

of permutations \pi \] such that for \] U \] ∈ \]Sm(U), U \] \] in U(i \] 1, ..., i_n; j_1, ..., j_n) and
\[ U_1 = U \cap \{1, ..., m_1\} \]
\[ \vdots \]
\[ U_n = U \cap \{m_1 + ... + m_{n-1} + 1, ..., m\}, \] (3.10)

(3.3) hold. For each \] U \] \] in U(i \] 1, ..., i_n; j_1, ..., j_n), the mapping
\[ j_U : \pi \mapsto (\alpha_U \circ \pi, \beta_V \circ \pi \circ \sigma_{i_1 + ... + i_n}), \quad \pi \in Sm(U) \] (3.11)

is a one-to-one correspondence between \]Sm(U) and \]S_{i_1 + ... + i_n} \times S_{j_1 + ... + j_n}. \]

Proof For any \] U \] \] in U(i \] 1, ..., i_n; j_1, ..., j_n), \] \] i_1 \] ∈ S_{i_1 + ... + i_n} \] and \] i_2 \] ∈ S_{j_1 + ... + j_n}, \] \] V = V_1 \cup ... \cup V_n \] is given by (3.4) and \] \pi \] \] in Sm is defined by \] \pi(i) = \alpha_U^{-1}(\pi_1(i)) \] for all \] i = 1, ..., i_1 + ... + i_n \] and \] \pi(\sigma_{i_1 + ... + i_n}(i)) = \beta_V^{-1}(\pi_2(i)) \] for all \] i = 1, ..., j_1 + ... + j_n, \] then \] (U, \pi_1, \pi_2) = (U_\pi, \alpha_U \circ \pi, \beta_V \circ \pi \circ \sigma_{i_1 + ... + i_n}) \] and because (3.8) and (3.4) hold and \] U = \{\pi(1), ..., \pi(i_1 + ... + i_n)\}, \] it follows that (3.1) hold. Consequently, \] \pi \] \] in Sm(i \] 1, ..., i_n; j_1, ..., j_n), so the map \] j \] is onto.

On the other hand, if \] \pi, \pi' \] \] in Sm, then \] U = U_n = U_{\pi'}, \alpha_U \circ \pi = \alpha_U \circ \pi' \] and \] \beta_V \circ \pi \circ \sigma_{i_1 + ... + i_n} = \beta_V \circ \pi' \circ \sigma_{i_1 + ... + i_n} \] imply \] \pi = \pi', \] so the map \] j \] is one-to-one as well.
The properties of the map \( j_U \) are obtained by observing that
\[
S_m(U) = j^{-1} \left( U \times S_{i_1 + \cdots + i_n} \times S_{j_1 + \cdots + j_n} \right)
\]
for \( U \in \mathcal{U}(i_1, \ldots, i_n; j_1, \ldots, j_n) \) and \( j_U = j \restriction S_m(U) \).

We can now prove

**Theorem 3.4** Let \( A_1, \ldots, A_n \) be bounded linear operators on a Banach space \( X \) and let \( \mu_1, \ldots, \mu_n \) be continuous measures on \([0, 1]\).
Also suppose that
\[
0 = t_0 < t_1 < t_2 < \cdots < t_{d-1} < t_d = 1 \tag{3.12}
\]
and \( I_k = [t_{k-1}, t_k] \) for \( k = 1, \ldots, d \). Let \( \mu_{j,k} = I_k.\mu_j \) for \( j = 1, \ldots, n \) and \( k = 1, \ldots, d \). Then for all nonnegative integers \( m_1, \ldots, m_n \) we have
\[
P_{\mu_1, \ldots, \mu_n}^{m_1, \ldots, m_n}(A_1, \ldots, A_n)
= \sum_{i_1 + \cdots + i_d = m_1} \frac{m_1!}{i_1! \cdots i_d!} \cdots \frac{m_n!}{i_1! \cdots i_d!} \times P_{\mu_{j_1}, \ldots, \mu_{j_d}}^{i_1, \ldots, i_d}(A_1, \ldots, A_n) \cdots P_{\mu_{j_1}, \ldots, \mu_{j_d}}^{i_1, \ldots, i_d}(A_1, \ldots, A_n).
\]

**Proof** Let \( t \in (0, 1) \) and let \( \mu_{j,1} = [0, t).\mu_j \) and \( \mu_{j,2} = [t, 1].\mu_j \) for \( j = 1, \ldots, n \). By induction, it suffices to take \( d = 2 \) and prove that
\[
P_{\mu_1, \ldots, \mu_n}^{m_1, \ldots, m_n}(A_1, \ldots, A_n)
= \sum_{i_1 + j_1 = m_1} \frac{m_1!}{i_1! j_1!} \cdots \frac{m_n!}{i_n! j_n!} P_{\mu_{j_1, \ldots, j_d}}^{i_1, \ldots, i_d}(A_1, \ldots, A_n) P_{\mu_{i_1, \ldots, i_d}}^{j_1, \ldots, j_d}(A_1, \ldots, A_n).
\]

The set \( \Delta_m(\pi) \) can be written up to a set of measure zero as
\[
\bigcup_{i_1 + j_1 = m_1} \Delta_m(\pi; i_1, \ldots, i_n; j_1, \ldots, j_n)
\]
in which the nonempty sets in the union are pairwise disjoint.

The integrals over the empty set are zero, so according to Lemma 3.1, we may write
\[
P_{\mu_1, \ldots, \mu_n}^{m_1, \ldots, m_n}(A_1, \ldots, A_n)
\]
\[ \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)}) (\mu_1^{m_1} \times \cdots \times \mu_n^{m_n}) (ds_1, \ldots, ds_m) = \sum_{\pi \in S_m} \sum_{i_1 + j_1 = m} \int_{\Delta_m(\pi; i_1, \ldots, i_n; j_1, \ldots, j_n)} C_{\pi(m)}(s_{\pi(m)}) C_{\pi(m-1)}(s_{\pi(m-1)}) \times \cdots \times C_{\pi(1)}(s_{\pi(1)}) (\mu_1^{m_1} \times \cdots \times \mu_n^{m_n}) (ds_1, \ldots, ds_m) = \sum_{i_1 + j_1 = m} \sum_{\pi \in S_m(i_1, \ldots, i_n; j_1, \ldots, j_n)} \int_{\Delta_m(\pi; i_1, \ldots, i_n; j_1, \ldots, j_n)} C_{\pi(m)}(s_{\pi(m)}) \times \cdots \times C_{\pi(1)}(s_{\pi(1)}) (\mu_1^{m_1} \times \cdots \times \mu_n^{m_n}) (ds_1, \ldots, ds_m). \] (3.13)

The last equality is valid because, according to (3.2), the permutation \( \pi \in S_m \) belongs to \( S_m(i_1, \ldots, i_n; j_1, \ldots, j_n) \) if and only if the open set \( \Delta_m(\pi; i_1, \ldots, i_n; j_1, \ldots, j_n) \) is non-empty.

Let

\[ C'_{i_1} : = \begin{cases} A_1 & \text{if } i \in \{1, \ldots, i_1\}, \\
A_2 & \text{if } i \in \{i_1 + 1, \ldots, i_1 + i_2\} \\
& \vdots \\
A_n & \text{if } i \in \{i_1 + \cdots + i_{n-1} + 1, \ldots, i_1 + \cdots + i_n\}, \end{cases} \] (3.14)

\[ C''_{j_1} : = \begin{cases} A_1 & \text{if } j \in \{1, \ldots, j_1\}, \\
A_2 & \text{if } j \in \{j_1 + 1, \ldots, j_1 + j_2\} \\
& \vdots \\
A_n & \text{if } j \in \{j_1 + \cdots + j_{n-1} + 1, \ldots, j_1 + \cdots + j_n\}, \end{cases} \] (3.15)

and suppose that \( \pi \in S_m(i_1, \ldots, i_n; j_1, \ldots, j_n) \) and \( U = U_1 \cup \cdots \cup U_n \) and \( V = V_1 \cup \cdots \cup V_n \) are given by (3.3) and (3.4). The enumerations \( \alpha_U \) of \( U \) and \( \beta_V \) of \( V \) are defined in the paragraph preceding (3.7).

Because \( \alpha_U(U_1) = \{1, \ldots, i_1\} \) and \( \alpha_U(U_k) = \{i_1 + \cdots + i_{k-1} + 1, i_1 + \cdots + i_{k-1} + i_k\} \) for \( k = 2, \ldots, n \), it follows that

\[ C_{\pi(i_1 + \cdots + i_n)} \cdots C_{\pi(1)} = C'_{\alpha_U \circ \pi(i_1 + \cdots + i_n)} \cdots C'_{\alpha_U \circ \pi(1)}. \]

Similarly, \( C_{\pi(m)} \cdots C_{\pi(i_1 + \cdots + i_n + 1)} = C''_{\beta_V \circ \pi(m)} \cdots C''_{\beta_V \circ \pi(i_1 + \cdots + i_n + 1)} \), so if we set \( \pi_1 = \alpha_U \circ \pi \) and \( \pi_2 = \beta_V \circ \pi \circ \sigma_{i_1 + \cdots + i_n} \), then according to (3.7), we have

\[ \int_{\Delta_m(\pi; i_1, \ldots, i_n; j_1, \ldots, j_n)} C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)}) (\mu_1^{m_1} \times \cdots \times \mu_n^{m_n}) (ds_1, \ldots, ds_m) \]
\[
\left( \int_{1 < s_{π2(1)} < \cdots < s_{π2(j_1 + \cdots + j_n)} < 1} C''_{π2(j_1 + \cdots + j_n)} \cdots C''_{π2(1)} \times \left( \prod_{j_1}^{i_1} \mu_{j_1} \prod_{j_2}^{i_2} \mu_{j_2} \right) (ds_1, \ldots, ds_{j_1 + \cdots + j_n}) \right) \\
\times \left( \int_{0 < s_{π1(1)} < \cdots < s_{π1(i_1 + \cdots + i_n)} < 1} C'_{π1(i_1 + \cdots + i_n)} \cdots C'_{π1(1)} \times \left( \prod_{i_1}^{j_1} \mu_{i_1} \prod_{i_2}^{j_2} \mu_{i_2} \right) (ds_1, \ldots, ds_{i_1 + \cdots + i_n}) \right) \\
= P_{j_1, \ldots, j_n} (A_1, \ldots, A_n) P_{i_1, \ldots, i_n} (A_1, \ldots, A_n). \tag{3.16}
\]

According to Lemma 3.3, the sum
\[
\sum_{π \in S_m(i_1, \ldots, i_n; j_1, \ldots, j_n)} \int_{Δm(π; i_1, \ldots, i_n; j_1, \ldots, j_n)} \cdots
\]

in (3.13) can be written as
\[
\sum_{U \in U(i_1, \ldots, i_n; j_1, \ldots, j_n)} \sum_{π \in S_m(U)} \int_{Δm(π; i_1, \ldots, i_n; j_1, \ldots, j_n)} \cdots.
\]

Let \( U \in U(i_1, \ldots, i_n; j_1, \ldots, j_n). \) Summing the expression (3.16) for \( \pi \in S_m(U) \) and noting that the map \( j_U : \pi \mapsto (π_1, π_2) \) defined in (3.11) is a one-to-one correspondence between \( S_m(U) \) and \( S_{i_1 + \cdots + i_n} \times S_{j_1 + \cdots + j_n}, \) we have
\[
\sum_{π \in S_m(U)} \int_{Δm(π; i_1, \ldots, i_n; j_1, \ldots, j_n)} C_{π(m)}(s_{π(m)}) \cdots C_{π(1)}(s_{π(1)})(\prod_{m_1}^{m_{i_1}} \mu_{m_{i_1}} \cdots \prod_{n_2}^{m_{j_2}} \mu_{m_{j_2}})(ds_1, \ldots, ds_m)
\]

\[
= \sum_{π_{1} \in S_{i_1 + \cdots + i_n}, π_{2} \in S_{j_1 + \cdots + j_n}} \left( \int_{1 < s_{π2(1)} < \cdots < s_{π2(j_1 + \cdots + j_n)} < 1} C''_{π2(j_1 + \cdots + j_n)} \cdots C''_{π2(1)} \times \left( \prod_{j_1}^{i_1} \mu_{j_1} \prod_{j_2}^{i_2} \mu_{j_2} \right) (ds_1, \ldots, ds_{j_1 + \cdots + j_n}) \right) \\
\times \left( \int_{0 < s_{π1(1)} < \cdots < s_{π1(i_1 + \cdots + i_n)} < 1} C'_{π1(i_1 + \cdots + i_n)} \cdots C'_{π1(1)} \times \left( \prod_{i_1}^{j_1} \mu_{i_1} \prod_{i_2}^{j_2} \mu_{i_2} \right) (ds_1, \ldots, ds_{i_1 + \cdots + i_n}) \right) \\
= P_{j_1, \ldots, j_n} (A_1, \ldots, A_n) P_{i_1, \ldots, i_n} (A_1, \ldots, A_n). \tag{3.17}
\]

It follows from (3.13), (3.17) and the observation that there are \( \frac{m_{i_k}}{l_k} \) subsets
\[
U_k \subset \{m_0 + \cdots + m_{k-1} + 1, \ldots, m_1 + \cdots + m_k\}
\]
such that \( \# U_k = i_k \) for each \( i_k = 0, \ldots, m_k \) and \( k = 1, \ldots, n \), that the bounded linear operator \( P_{\mu_1, \ldots, \mu_n}^{m_1, \ldots, m_n} (A_1, \ldots, A_n) \) equals

\[
\sum_{i_1 + j_1 = m_1} \frac{m_1!}{i_1! j_1!} \ldots \frac{m_n!}{i_n! j_n!} P_{\mu_{1,1}, \ldots, \mu_{n,1}}^{j_1, \ldots, j_n} (A_1, \ldots, A_n) P_{\mu_{1,1}, \ldots, \mu_{n,1}}^{i_1, \ldots, i_n} (A_1, \ldots, A_n),
\]

as was to be proved. Here we have adopted the convention that an integral with respect to the zero'th power \( \mu^0 \) of a measure \( \mu \) is omitted.

\[\square\]

**Corollary 3.5** With the same notation as Theorem 3.4, we have

\[ T_{\mu_1, \ldots, \mu_n} (e^{\hat{A}_1 + \cdots + \hat{A}_n}) = T_{\mu_1, d, \ldots, \mu_n, d} (e^{\hat{A}_1 + \cdots + \hat{A}_n}) \ldots T_{\mu_1, 1, \ldots, \mu_n, 1} (e^{\hat{A}_1 + \cdots + \hat{A}_n}). \quad (3.18) \]

**Proof** From formula (1.10), we have

\[
T_{\mu_1, d, \ldots, \mu_n, d} (e^{\hat{A}_1 + \cdots + \hat{A}_n}) \ldots T_{\mu_1, 1, \ldots, \mu_n, 1} (e^{\hat{A}_1 + \cdots + \hat{A}_n})
\]

\[
= \left( \sum_{i_1 d, \ldots, i_n d = 0}^{\infty} \frac{1}{i_1 d! \ldots i_n d!} P_{\mu_{1, d}, \ldots, \mu_{n, d}}^{i_1 d, \ldots, i_n d} (A_1, \ldots, A_n) \right)
\]

\[
\ldots \left( \sum_{i_1 1, \ldots, i_n 1 = 0}^{\infty} \frac{1}{i_1 1! \ldots i_n 1!} P_{\mu_{1, 1}, \ldots, \mu_{n, 1}}^{i_1 1, \ldots, i_n 1} (A_1, \ldots, A_n) \right)
\]

\[
= \sum_{m_1 = 0}^{\infty} \ldots \sum_{m_n = 0}^{\infty} \frac{1}{m_1! \ldots m_n!} \left[ \sum_{i_1 1 + \cdots + i_n d = m_1}^{m_1!} \frac{m_1!}{i_1 1! \cdots i_1 d! \ldots i_n 1! \cdots i_n d!} \right]
\]

\[
\times P_{\mu_{1, 1}, \ldots, \mu_{n, 1}}^{i_1 1, \ldots, i_n 1} (A_1, \ldots, A_n) \ldots P_{\mu_{1, d}, \ldots, \mu_{n, d}}^{i_1 d, \ldots, i_n d} (A_1, \ldots, A_n)
\]

and

\[ T_{\mu_1, \ldots, \mu_n} (e^{\hat{A}_1 + \cdots + \hat{A}_n}) = \sum_{m_1 = 0}^{\infty} \ldots \sum_{m_n = 0}^{\infty} \frac{1}{m_1! \ldots m_n!} P_{\mu_{1, 1}, \ldots, \mu_{n, n}}^{m_1, \ldots, m_n} (A_1, \ldots, A_n). \]

According to Theorem 3.4, the two multiple series are identical. This proves formula (3.18).

\[\square\]

If we allow the times \( \langle t_j \rangle_{j=1}^d \) to vary, then an argument similar to that of the proof of Theorem 3.4 gives the following perturbation result.

**Theorem 3.6** Let \( A_1, \ldots, A_n, B \) be bounded linear operators on a Banach space \( X \) and let \( \mu_1, \ldots, \mu_n, \nu \) be continuous measures on \([0, 1]\).
Then for all nonnegative integers \( m_1, \ldots, m_n, k \), we have
\[
P^{m_1, \ldots, m_n, k}_{\mu_1, \ldots, \mu_n, v}(A_1, \ldots, A_n, B) = k! \times \sum_{i_1 + \ldots + i_k = m_1} \frac{m_1!}{i_1! \cdots i_k!} \cdots \sum_{i_n + \ldots + i_k = m_n} \frac{m_n!}{i_n! \cdots i_k!} 
\]
\[
\times \int_{\Delta_k} P^{[k_1, \ldots, k_n]}_{[\mu_1, \ldots, \mu_n]}(A_1, \ldots, A_n) B P^{[1, \ldots, i_{n-1}, \ldots, i_{k-1}]}_{[\mu_1, \ldots, \mu_n]}(A_1, \ldots, A_n) \cdots B P^{[0, \ldots, i_{n-1}, \ldots, i_{k-1}]}_{[\mu_1, \ldots, \mu_n]}(A_1, \ldots, A_n) v^k(ds_1, \ldots, ds_k). 
\] (3.19)

Proof of Theorem 2.3 First we expand the integrand of each term in the sum on the right hand side of (2.4). We put \( s_0 = 0, s_{k+1} = 1 \) and
so that
\[
\mu_{j, \ell} = [s_{\ell-1}, s_{\ell}), \mu_j, \quad j = 1, \ldots, n, \quad \ell = 1, \ldots, k + 1
\]
\[
T_{[\mu_1, \ldots, \mu_n]}(e^{\tilde{\lambda}_1 + \cdots + \tilde{\lambda}_n}) B T_{[\mu_1, \ldots, \mu_n]}(e^{\tilde{\lambda}_1 + \cdots + \tilde{\lambda}_n}) \cdots
\]
\[
\cdots T_{[\mu_{n, \mu_{n-1}, \ldots, \mu_1]}(e^{\tilde{\lambda}_1 + \cdots + \tilde{\lambda}_n}) B^T_{[\mu_{n, \mu_{n-1}, \ldots, \mu_1]}(e^{\tilde{\lambda}_1 + \cdots + \tilde{\lambda}_n})}
\]
\[
= \left( \sum_{i_1 + \ldots + i_{n-1} = 0}^\infty \frac{1}{i_1! \cdots i_{n-1}!} P^{[1, \ldots, i_{n-1}, \ldots, i_{k-1}]}_{[\mu_1, \ldots, \mu_n]}(A_1, \ldots, A_n) \right) B
\]
\[
\cdots B \left( \sum_{i_1 + \ldots + i_{n-2} = 0}^\infty \frac{1}{i_1! \cdots i_{n-2}!} P^{[1, \ldots, i_{n-2}, \ldots, i_{k-1}]}_{[\mu_1, \ldots, \mu_n]}(A_1, \ldots, A_n) \right)
\]
\[
\times B \left( \sum_{i_1 + \ldots + i_0 = 0}^\infty \frac{1}{i_1! \cdots i_0!} P^{[1, \ldots, i_{n-1}, \ldots, i_{k-1}]}_{[\mu_1, \ldots, \mu_n]}(A_1, \ldots, A_n) \right)
\]
\[
= \sum_{m_1 = 0}^\infty \sum_{m_2 = 0}^\infty \frac{m_1!}{i_1! \cdots i_{k-1}!} \sum_{m_2 = 0}^\infty \frac{m_2!}{i_0! \cdots i_{k-1}!} \cdots \frac{m_n!}{i_0! \cdots i_{k-1}!}
\]
\[
\times P^{[1, \ldots, i_{n-1}, \ldots, i_{k-1}]}_{[\mu_1, \ldots, \mu_n]}(A_1, \ldots, A_n) B P^{[1, \ldots, i_{n-1}, \ldots, i_{k-1}]}_{[\mu_1, \ldots, \mu_n]}(A_1, \ldots, A_n) \cdots B P^{[1, \ldots, i_{n-1}, \ldots, i_{k-1}]}_{[\mu_1, \ldots, \mu_n]}(A_1, \ldots, A_n).
\]

By Theorem 3.6, the right hand side of (2.4) is therefore equal to
\[
\sum_{k=0}^\infty \sum_{m_1 = 0}^\infty \cdots \sum_{m_n = 0}^\infty \frac{1}{m_1! \cdots m_n! k!} P^{m_1, \ldots, m_n, k}_{\mu_1, \ldots, \mu_n, v}(A_1, \ldots, A_n, B) = T_{[\mu_1, \ldots, \mu_n, v]}(e^{\tilde{\lambda}_1 + \cdots + \tilde{\lambda}_n + \tilde{B}}).
\]

\( \square \)
4 Decomposition as an aid to Disentangling: Examples

In the examples below we begin with 2 (or 3) operator-measure pairs \((A, \mu), (B, \nu)\) (and sometimes \((C, \eta)\)) in order to simplify notation. In fact, these things can all be done starting with any finite number of such pairs. We assume throughout that \(\mu, \nu\) contain the supports of the measures \(\mu\). Sometimes \(\mu, \nu\) be restated as corollaries. Once we can decompose monomials in the disentangling algebra, it is straightforward to treat polynomials or analytic functions that belong to an appropriate disentangling algebra.

The support of a Borel measure \(\lambda\) on \([0, 1]\) is written as \(S(\lambda)\). We write \(U \leq V\) for two subsets \(U, V\) of \([0, 1]\) if \(u \leq v\) for all \(u \in U\) and \(v \in V\).

We start with an example which is rather simple but illustrates some key points.

Example 4.1 Let \(0 < a < 1\). We take \(\mu = \mu_1 + \mu_2\) and \(\nu = v_1 + v_2\) where

\[
\mu_1 := [0, a]. \mu, \quad \mu_2 := [a, 1]. \mu, \quad v_1 := [0, a]. \nu \quad \text{and} \quad v_2 := [a, 1]. \nu.
\]

We will use Theorem 3.4 with \(n = 2\) (or Corollary 2.2) to decompose the disentangling \(P_{\mu, \nu}(A, B)\) for \(A, B \in \mathcal{L}(X)\). First, we will carry out the calculation in (4.1) below; next we will make some remarks about it and finally write out a specific case. According to Theorem 3.4

\[
P_{\mu, \nu}(A, B) = \sum_{i_1 + j_1 = 1} P_{\mu_1, v_1}(A_1, A_2)P_{\mu_2, v_2}(A_1, A_2).
\]

(4.1)

The coefficient from each of the four terms on the right hand side of (4.1) equals one because the factorials involved are either 0! or 1! in this case and we have

\[
P_{\mu, \nu}(A, B) = P_{\mu_1, v_1}(A, B) + P_{\mu_2, v_2}(A, B) + P_{\mu_1, v_1}(A, B) + P_{\mu_2, v_2}(A, B)\]

\[
= I. P_{\mu_1, v_1}(A, B) + P_{\mu_1, v_1}(A, B) + P_{\mu_2, v_2}(A, B) + P_{\mu_2, v_2}(A, B).
\]

(4.2)

Now draw the square \([0, 1]^2\) and place \(a\) along both the horizontal and vertical axes. Consider the four rectangles \([0, a] \times [a, 1], [a, 1] \times [0, a], [0, a]^2\) and \([a, 1]^2\). The rectangles contain the supports of the measures \(\mu_1 \times \mu_2, \mu_2 \times \mu_1, \mu_1 \times \nu_1\) and \(\mu_2 \times \nu_2\), respectively. The rectangle \([0, a] \times [a, 1]\) lies above the diagonal \(s_2 = s_1\) and so the operator ordering in that rectangle is \(BA\). The coefficient is \(\mu_1([0, a]) \nu_2([a, 1]) = ||\mu_1|| \cdot ||\nu_2||\). Thus the second term of the right hand side of the equality (4.2) should equal \(||\mu_1|| \cdot ||\nu_2|| BA\), as is the case.

In a similar fashion, the rectangle \([a, 1] \times [0, a]\) below the diagonal and so the operator ordering is \(AB\) and the third term of the right hand side of the equality (4.2) is \(||\mu_2|| \cdot ||\nu_1|| AB\).
Nothing more needs to be done with these two terms except that one needs to know one of \( \mu_1([0, a]) \) or \( \mu_2([a, 1]) \) and one of \( \nu_1([0, a]) \) or \( \nu_2([a, 1]) \) in order to calculate the coefficients. (This is a point that we shall usually not be concerned with.)

It remains to calculate the first and fourth terms of the right hand side of the equality (4.2). We discuss only the first one \( P_{\mu_1, \nu_1}^{1,1}(A, B) \) since \( P_{\mu_2, \nu_2}^{1,1}(A, B) \) is treated in much the same manner. By Definition 1.1, we have

\[
P_{\mu_1, \nu_1}^{1,1}(A, B) = (\mu_1 \times \nu_1)((s_1, s_2) : 0 < s_1 < s_2 < a)BA
+ (\mu_1 \times \nu_1)((s_1, s_2) : 0 < s_2 < s_1 < a)AB
= \left( \int_0^a \nu_1([s_1, a]) \mu_1(ds) \right) BA + \left( \int_0^a \nu_1([0, s_1]) \mu_1(ds) \right) AB. \quad (4.4)
\]

Now there are many cases where the last line of (4.4) can be explicitly calculated. For example, if \( \mu_1 \) and \( \nu_1 \) are absolutely continuous with respect to Lebesgue measure and both have polynomial density functions, the calculation can be carried out. In that case, even with higher powers, say \((m, n)\) instead of \((1, 1)\), the calculation can be done at least in principle; however, it will get extremely tedious very fast as the powers grow.

We finish our discussion of Example 4.1 with a simple concrete case. Take \( a = \frac{1}{2} \) and let \( \mu \) be Lebesgue measure on \([0, 1]\). Then \( \mu_1 \) and \( \mu_2 \) are Lebesgue measure on \([0, 1/2]\) and \([1/2, 1]\), respectively. Take \( \nu \) to be twice Lebesgue measure on \([0, 1/2]\) and the zero measure on \([1/2, 1]\). Then \( \nu_1 = \nu \) on \([0, 1/2]\) and \( \nu_2 = 0 \) on \([1/2, 1]\). Consider the right hand side of the equality (4.3). Since \( \nu_2 = 0, \mu_2 \times \nu_2 = 0 \) and \( \|\nu_2\| = 0 \). It follows that the third, fifth and sixth terms in the sum equal zero. The fourth term is easily seen to be \( \frac{1}{2} \cdot 1.AB \) and the sum of the first and second terms is \( \frac{1}{4}BA + \frac{1}{4}AB \). Thus, in this specific case,

\[
P_{\mu, \nu}^{1,1}(A, B) = \frac{1}{2}BA + \frac{1}{4}BA + \frac{1}{4}AB = \frac{1}{4}BA + \frac{3}{4}AB. \quad (4.5)
\]

According to [4, (2.29)], we have

\[
P_{\mu, \nu}^{1,1}(A, B) = pBA + (1 - p)AB, \quad p = (\mu \times \nu)((s_1 < s_2)).
\]

In the case under consideration \( p = \frac{1}{4} \), so we obtain agreement with (4.5).

**Example 4.2** Let \( \mu_1 = [0, a].\), \( \mu_2 = [a, 1].\), \( \nu_1 = [0, a].\), \( \nu_2 = [a, 1].\) as in Example 4.1. By Corollary 2.2 or Theorem 3.4,

\[
P_{\mu, \nu}^{m_1, m_2}(A, B) = \sum_{i_1 + j_1 = m_1} \sum_{i_2 + j_2 = m_2} \frac{m_1!}{i_1!j_1!} \frac{m_2!}{i_2!j_2!} P_{\mu_1, \nu_1}^{j_1, j_2}(A, B) P_{\mu_2, \nu_2}^{i_1, i_2}(A, B). \quad (4.6)
\]

Now we make an extra assumption on the supports of the measure:

\[
0 \leq S(\mu_1) \leq S(\nu_1) \leq a \leq S(\mu_2) \leq S(\nu_2) \leq 1. \quad (4.7)
\]

Since \( S(\mu_1) \leq S(\nu_1) \) and \( S(\mu_2) \leq S(\nu_2) \), we can apply [3, Corollary 4.3] to conclude that \( P_{\mu_1, \nu_1}^{i_1, j_1}(A, B) = (\|\nu_1\|B)^{j_1}(\|\mu_1\|A)^{i_1} \) and \( P_{\mu_2, \nu_2}^{i_2, j_2}(A, B) = (\|\nu_2\|B)^{j_2}(\|\mu_2\|A)^{i_2} \) and so, from (4.6),

\[
P_{\mu, \nu}^{m_1, m_2}(A, B) = \sum_{i_1 + j_1 = m_1} \sum_{i_2 + j_2 = m_2} \frac{m_1!}{i_1!j_1!} \frac{m_2!}{i_2!j_2!} (\|\nu_2\|B)^{j_2}(\|\mu_2\|A)^{i_2} (\|\nu_1\|B)^{j_1}(\|\mu_1\|A)^{i_1} \). \quad (4.8)
\]
There are three further choices for arranging the supports before and after $a$:

(i) Reverse the ordering of the supports on both sides of $a$.

(ii) Change the order on the left of $a$ but not on the right.

(iii) Change the order on the right of $a$ but not on the left.

For example, in the third case we obtain

$$ P_{\mu, v}^{m_1, m_2} (A, B) = \sum_{i_1 + j_1 = m_1} \frac{m_1!}{i_1! j_1!} \frac{m_2!}{i_2! j_2!} (\| \mu_2 \| A)^{i_1} (\| v_2 \| B)^{j_1} (\| v_1 \| B)^{i_2} (\| \mu_1 \| A)^{j_1} \mu, v \in P_{\mu, v}^{m_1, m_2} (A, B)$$

$$= \sum_{i_1 + j_1 = m_1} \frac{m_1!}{i_1! j_1!} \frac{m_2!}{i_2! j_2!} (\| \mu_2 \| A)^{i_1} (\| v_2 \| B)^{j_1} (\| v_1 \| B)^{i_2} B^{m_2} (\| \mu_1 \| A)^{j_1}. \quad (4.9) $$

We look at one more variation of this family of examples. Suppose that the supports of $\mu_1$ and $v_1$ are ordered by $0 \leq S(\mu_1) \leq S(v_1) \leq a$, but that we have no such information on the interval $[a, 1]$. Then

$$ P_{\mu, v}^{m_1, m_2} (A, B) = \sum_{i_1 + j_1 = m_1} \frac{m_1!}{i_1! j_1!} \frac{m_2!}{i_2! j_2!} \frac{m_3!}{i_3! j_3! k_1! k_2! k_3!} P_{\mu_2, v_2}^{i_1, j_1, k_1} (A, B) (\| v_1 \| B)^{i_2} (\| \mu_1 \| A)^{j_1}. \quad (4.10) $$

Example 4.3 Here we consider the case of three operator-measure pairs $(A, \mu)$, $(B, \nu)$, $(C, \eta)$ where $\mu$, $\nu$ and $\eta$ are measures on $[0, 1]$ which satisfy our usual conditions. We partition $[0, 1]$ into 3 subintervals, $0 < a < b < 1$. Actually, it is often useful to decide on the number of subintervals and their placement after one knows the nature of the measures and the location of their supports. We let

$$ \mu_1 := [0, a], \mu, \quad \mu_2 := [a, b], \mu \quad \text{and} \quad \mu_3 := [b, 1], \mu; $$

$$ v_1 := [0, a], v, \quad v_2 := [a, b], v \quad \text{and} \quad v_3 := [b, 1], v; $$

$$ \eta_1 := [0, a], \eta, \quad \eta_2 := [a, b], \eta \quad \text{and} \quad \eta_3 := [b, 1], \eta. $$

Also, we assume for now that each of $\mu$, $\nu$, $\eta$ has a nonzero part of its support in each of the three intervals $[0, a]$, $[a, b]$, $[b, 1]$.

Applying Theorem 3.4 again, we have

$$ P_{\mu_1, \nu_1, \eta_1}^{m_1, m_2, m_3} (A, B, C) $$

$$= \sum_{i_1 + j_1 + k_1 = m_1} \frac{m_1!}{i_1! j_1! k_1!} \frac{m_2!}{i_2! j_2! k_2!} \frac{m_3!}{i_3! j_3! k_3!} \times P_{\mu_2, \nu_2, \eta_2}^{i_1, j_1, k_1} (A, B, C) P_{\mu_3, \nu_3, \eta_3}^{i_2, j_2, k_2} (A, B, C) P_{\mu_4, \nu_4, \eta_4}^{i_3, j_3, k_3} (A, B, C). \quad (4.10) $$

Now turning to special cases, we assume first that

$$ S(\mu_1) \leq S(v_1) \leq S(\eta_1) \quad \text{on} \quad [0, a], $$
An appeal to [3, Corollary 4.3] shows that

\[
P_{\mu, \nu, \eta}^{m_1, m_2, m_3}(A, B, C) = \sum_{i_1 + j_1 + k_1 = m_1} \frac{m_1!}{i_1! j_1! k_1!} \sum_{i_2 + j_2 + k_2 = m_2} \frac{m_2!}{i_2! j_2! k_2!} \sum_{i_3 + j_3 + k_3 = m_3} \frac{m_3!}{i_3! j_3! k_3!} \left( \| \eta_3 \| C \right)^{k_3} \left( \| \nu_3 \| B \right)^{k_2} \left( \| \mu_3 \| A \right)^{k_1} \times \left( \| \eta_2 \| C \right)^{j_3} \left( \| \nu_2 \| B \right)^{j_2} \left( \| \mu_2 \| A \right)^{j_1} \left( \| \eta_1 \| C \right)^{i_3} \left( \| \nu_1 \| B \right)^{i_2} \left( \| \mu_1 \| A \right)^{i_1}.
\] (4.11)

Next suppose that the supports are ordered within each of the three intervals \([0, a], [a, b]\) and \([b, 1]\) but not necessarily as above. Then there are 3! possible orderings of the supports in each interval and so \((3!)^3 = 216\) such orderings in all. Taking into account the fact that one or more of the 9 nonnegative integers \(i_1, i_2, \ldots, k_3\) can equal zero, we see that there are a large number of operator orderings that are produced in this situation. (Of course, the coefficients of the terms on the right hand side of (4.11) are varying as well.)

We consider one more variation of Example 4.3 above. We take the assumptions on the support of the measures on \([0, a]\) and on \([a, b]\) to be exactly as described just before formula (4.11). However, we change the assumptions on the supports of \(\mu_3, \nu_3\) and \(\eta_3\) as follows: let \([\gamma, \delta]\) be an interval which is internal to \([b, 1]\) and assume that the supports of \(\nu_3\) and \(\eta_3\) are both contained in \([\gamma, \delta]\) but that \(S(\mu_3) \subset [b, \gamma] \cup [\delta, 1]\). Then the theorem on extracting a linear factor [5, Theorem 2.1 or Theorem 2.3] allows us to write

\[
P_{\mu, \nu, \eta}^{k_1, k_2, k_3}(A, B, C) = P_{\mu, \nu}^{k_1, 1}(A, K_0)
\]

where \(K_0 = P_{\nu_3, \eta_3}^{k_2, k_3}(B, C)\) and \(\beta_0\) is any continuous probability measure on \([\gamma, \delta]\). By decomposing \([b, 1]\) into three intervals \([b, 1] = [b, \gamma] \cup [\gamma, \delta] \cup [\delta, 1]\), we can disentangle \(P_{\mu_3, \nu_3}^{k_2, k_3}(A, K_0)\) by a further application of Theorem 3.4. To fully disentangle \(P_{\mu, \nu, \eta}^{m_1, m_2, m_3}(A, B, C)\) in the present case requires the calculation of \(K_0\) by some earlier method or by definition, as was discussed in Example 4.1.

In the following example, we look at two cases where not all of the operators involved appear in every interval. Although we do not discuss situations where a complete disentangling is achieved, a considerable simplification of the disentangling formulae (4.6) and (4.10) is obtained.

**Example 4.4** Let \(\mu\) and \(\nu\) satisfy the usual conditions on \([0, 1]\) and let \(a \in (0, 1)\). We write \(\mu = \mu_1 + \mu_2\) where \(S(\mu_1) \subset [0, a]\) and \(\mu_2\) is the zero measure on \([a, 1]\). Also we take \(v = v_1 + v_2\) where \(v_1 := [0, a], \nu\) and \(v_2 := [a, 1], \nu\). By (4.6) we have

\[
P_{\mu, \nu}^{m_1, m_2}(A, B) = \sum_{i_1 + j_1 + m_1 = 1} \frac{m_1!}{i_1! j_1!} \frac{m_2!}{i_2! j_2!} P_{\mu_1, \nu_1}^{i_1, j_1}(A, B) P_{\mu_2, \nu_2}^{i_2, j_2}(A, B) = \sum_{i_2 + j_2 + m_2 = 1} \frac{m_1!}{i_2! j_2!} \frac{m_2!}{m_1! i_2! j_2!} P_{\mu_1, \nu_1}^{i_1, j_1}(A, B) P_{\mu_2, \nu_2}^{i_2, j_2}(A, B).
\]
write briefly μ

\[ \text{let } \mu, \nu, \eta \text{ satisfy the usual conditions on } [0, 1] \] and let \( 0 < a < b < 1 \). We write \( \mu = \mu_1 + \mu_2 + \mu_3 \) where \( S(\mu_1) \subseteq [0, a] \), \( S(\mu_3) \subseteq [b, 1] \) and \( \mu_2 \) is the zero measure on \([a, b] \). In a similar manner, we write \( \nu = \nu_1 + \nu_2 + \nu_3 \) where \( S(\nu_1) \subseteq [0, a] \), \( S(\nu_2) \subseteq [a, b] \) and \( \nu_3 \) is the zero measure on \([b, 1] \). Finally, let \( \eta = \eta_1 + \eta_2 + \eta_3 \) where \( S(\eta_1) \subseteq [0, a] \), \( S(\eta_3) \subseteq [b, 1] \) and \( \eta_2 \) is the zero measure on \([a, b] \). Hence, we can write briefly \( \mu = \mu_1 + 0 + \mu_3 \), \( \nu = \nu_1 + \nu_2 + 0 \) and \( \eta = \eta_1 + 0 + \eta_3 \). According to (4.10) we have

\[ P_{\mu, \nu, \eta}^{m_1, m_2, m_3}(A, B, C) = \sum_{i_1^1 + k_1 = m_1} m_1! \sum_{i_2^1 + j_2 = m_2} m_2! \sum_{i_3^1 + k_3 = m_3} m_3! \]

\[ \times P_{\mu_1, \nu_1, \eta_1}^{k_1, k_3}(A, B, C) \]

and

\[ \times P_{\mu_1, \nu_1, \eta_1}^{k_1, k_3}(\|\nu_2\| B) \]

\[ + \sum_{i_1^1 + k_1 = m_1} m_1! \sum_{i_2^1 + j_2 = m_2} m_2! \sum_{i_3^1 + k_3 = m_3} m_3! \]

\[ \times P_{\mu_1, \nu_1, \eta_1}^{k_1, k_3}(A, C)(\|\nu_2\| B) \]

\[ P_{\mu_1, \nu_1, \eta_1}^{i_1, i_2, i_3}(A, B, C) \]

(4.12)

Note that the last expression of the operator product (4.12) involves three operator-measure pairs whereas the first expression involves two such pairs. The middle expression involves only one such pair, so it is trivial to disentangle.

We have seen in Corollary 3.5 that the exponential functions and disentangling maps fit together well. The following family of related examples can be completely disentangled.

**Example 4.5** Let \( \mu_j \), \( \nu_j \) and \( \eta_j \), \( j = 1, \ldots, k \) be finite, continuous measures on \([0, 1] \) and let

\[ \mu = \sum_{j=1}^{k} \mu_j, \quad \nu = \sum_{j=1}^{k} \nu_j, \quad \eta = \sum_{j=1}^{k} \eta_j. \]  

(4.13)

Furthermore, suppose that

\[ 0 = a_1 < b_1 < c_1 < a_2 < b_2 < c_2 < \cdots < a_k < b_k < c_k < a_{k+1} = 1 \]

and \( S(\mu_j) \subseteq [a_j, b_j] \), \( S(\nu_j) \subseteq [b_j, c_j] \) and \( S(\eta_j) \subseteq [c_j, a_{j+1}] \) for all \( j = 1, \ldots, k \). Finally, let \( A, B, C \in \mathcal{L}(X) \). Then

\[ T_{\mu, \nu, \eta}^{\tilde{A} + \tilde{B} + \tilde{C}} = e^{\eta_k ||C e^{v_k} \| B e^{\mu_k} ||A} \cdots e^{\nu_1 ||C e^{v_1} \| B e^{\mu_1} ||A.} \]
Proof Break the interval $[0, 1]$ into $k$ pieces: $[a_1, a_2] \cup [a_2, a_3] \cup \cdots \cup [a_k, a_{k+1}]$. First using Corollary 3.5 and then the ordering result [3, Proposition 4.5] $k$ times, we obtain

$$T_{\mu, \nu, \eta}(e^{\hat{A} + \hat{B} + \hat{C}}) = T_{\mu_k, \nu_k, \eta_k}(e^{\hat{A} + \hat{B} + \hat{C}}) \cdots T_{\mu_1, \nu_1, \eta_1}(e^{\hat{A} + \hat{B} + \hat{C}}) = e^{\eta_k ||C||} e^{\nu_k ||B||} e^{\mu_k ||A||} \cdots e^{\eta_1 ||C||} e^{\nu_1 ||B||} e^{\mu_1 ||A||}.$$

□

There are several obvious variations of this example. Within each triple $\mu_j, \nu_j, \eta_j$, the supports of these three measures can be permuted in $3!=6$ ways. This yields $(3!)^k$ different disentanglings. Also, we can have any finite number, say $\ell$, of operator-measure pairs, so by similar reasoning, we get $(\ell!)^k$ different disentanglings.

We finish with two simple specific examples illustrating perturbation series coming from Theorem 2.3.

Example 4.6 We adopt the assumptions of Theorem 2.3. Let $A_1, \ldots, A_n$ and $B$ be operators from $\mathcal{L}(X)$ with associated finite, continuous measures $\mu_1, \ldots, \mu_n$ and $\nu$ on $\mathcal{B}[0, 1]$.

We assume in the first case that $\mu = \mu_1 = \cdots = \mu_n$. Then using Theorem 2.3 and the first example of a disentangling given in Sect. 1, we can write

$$T_{\mu, \ldots, \mu, \nu}(e^{\hat{A}_1 + \cdots + \hat{A}_n + \hat{B}}) = \sum_{k=0}^{\infty} \int_{\Delta_k} e^{\mu([s_{k,1},1])} A_n \cdots e^{\mu([s_{k-1,1}, s_k])} A_1 B \times e^{\mu([0,s_1])} A_n \cdots e^{\mu([0,s_1])} A_1 \nu^k(ds_1, \ldots, ds_k).$$

We do not assume in the second example that $\mu_1 = \cdots = \mu_n$ but rather that the supports of $\mu_1, \ldots, \mu_n$ are ordered. There are $n!$ possible orders; we consider only the order $S(\mu_1) \leq S(\mu_2) \leq \cdots \leq S(\mu_n)$, then by Corollary 3.5 and the ordering result [3, Proposition 4.5], we have

$$T_{\mu_1, \ldots, \mu_n, \nu}(e^{\hat{A}_1 + \cdots + \hat{A}_n + \hat{B}}) = \sum_{k=0}^{\infty} \int_{\Delta_k} e^{\mu_n([s_{k,1},1])} A_n \cdots e^{\mu_2([s_{k,1},1])} A_2 e^{\mu_1([s_{k,1},1])} A_1 B \times e^{\mu([0,s_1])} A_n \cdots e^{\mu([0,s_1])} A_1 \nu^k(ds_1, \ldots, ds_k).$$

References