Feynman’s Operational Calculus with Brownian Time-Ordering

Brian Jefferies*
School of Mathematics, The University of New South Wales, NSW 2052 Australia

Abstract

We consider Feynman’s operational calculus for two bounded linear operators on Hilbert space with Lebesgue measure as one time-ordering measure and Brownian motion replacing the second time-ordering measure. A representation for solutions of linear stochastic differential equations in Hilbert space is obtained.

2000 Mathematics Subject Classification: Primary 47A60; Secondary 47A13, 47N50, 60H25.

Key words and phrases: Functional calculus, disentangling, multiple stochastic integral.

1 Introduction

The author and Jerry Johnson studied a family of functional calculi for bounded linear operators $A_1, \ldots, A_n$ acting on a Banach space $X$ in a series of papers [7, 8, 9]. Each functional calculus is determined by $n$ continuous probability measures $\mu_1, \ldots, \mu_n$ defined on $\mathbb{B}([0,1])$, the Borel class of $[0,1]$. The time-ordering measures $\mu_1, \ldots, \mu_n$ determine an operational calculus or disentangling map $T_{\mu_1, \ldots, \mu_n}$ from a commutative Banach algebra $\mathbb{D}(A_1, \ldots, A_n)$ of analytic functions into the noncommutative Banach algebra $L(X)$, see [7]. For a function $f$ belonging to the disentangling algebra $\mathbb{D}(A_1, \ldots, A_n)$, the bounded linear operator $f_{\mu_1, \ldots, \mu_n}(A_1, \ldots, A_n) := T_{\mu_1, \ldots, \mu_n}f$ represents the function $f$ of the (constant) operator valued functions $A_j(t) := A_j$, $0 \leq t \leq 1$, after disentangling with respect to the time-ordering measures $\mu_1, \ldots, \mu_n$. A major motivation for developing an operational calculus is for representing solutions of evolution equations. For example, if $\lambda$ denotes Lebesgue measure on $[0,1]$, then

$$e^{A+B}_{\lambda,\lambda} = e^{A+B} = e^{tA} + \sum_{n=1}^{\infty} \int_0^1 \cdots \int_0^1 e^{(1-s_n)A}B e^{(s_n-s_{n-1})A} \cdots e^{(s_2-s_1)A} B e^{s_1A} ds_1 \cdots ds_n$$

(1.1)

*E-mail address: b.jefferies@unsw.edu.au
is the well known perturbation series expansion for the exponential of the sum of bounded linear operators $A$ and $B$. A similar result holds when $\lambda$ is replaced by any continuous Borel probability measure on the unit interval, see [8, Corollary 5.3].

Suppose that $A$ and $B$ are \textit{commuting} bounded linear operators acting on a Banach space $E$ and $W$ is a Brownian motion process. A solution of the linear operator valued stochastic differential equation
\begin{equation}
  dX_t + AX_t dt = BX_t dW_t
\end{equation}
in $\mathcal{L}(E)$, can be written as $X_t = \exp(-t(A + B^2/2) + BW_t)$, or,
\begin{equation}
  X_t = \exp \left[ -\int_0^t \left( A + \frac{1}{2} B^2 \right) ds + \int_0^t B dW_s \right];
\end{equation}
for the proof, it suffices to apply Itô’s formula scalarly. Formula (1.3) suggests that by taking $f(z_1, z_2) = e^{z_1 + z_2}$, we ought to able to write the solution of equation (1.2) as
\begin{equation}
  X_t = e^{-A+B}_{dt,dW_t} := f_{dt,dW_t}(-A,B)
\end{equation}
by using Feynman’s operational calculus with time ordering “measures” $(dt, dW_t)$ for the pair $(-A, B)$ of bounded linear operators, even if they do not commute.

G.W. Johnson and G. Kallianpur [11] have represented $X$ by the stochastic Dyson series
\begin{equation}
  X_t = e^{-tA} + \sum_{n=1}^{\infty} \int_0^t \int_0^{s_n} \cdots \int_0^{s_1} e^{-(t-s_n)A} B_{s_n} \cdots e^{-(s_2-s_1)A} B_{s_1} e^{-s_1A} dW_{s_1} \cdots dW_{s_n}
\end{equation}
with respect to time-ordered operator valued multiple Wiener integrals in the case that $-A$ is the generator of a $C_0$-contraction semigroup on Hilbert space, $B$ is an operator valued function of time and $\int_0^T \|B_s\|^2 ds < \infty$. Comparison with the perturbation expansion (1.1) above reveals the connection with the expression $e^{-A+B}_{dt,dW_t}$ suggested by Feynman’s operational calculus. So, we are seeking a \textit{stochastic functional calculus} $f \mapsto f_{dt,dW_t}(-A,B)$ based on time-ordering with respect to white noise $dW_t$, which will enable us to make sense of $e^{-A+B}_{dt,dW_t}$ even if $A$ and $B$ are unbounded linear operators. The present note is a first step in that direction.

In Section two we revisit the notation and definition of Feynman’s operational calculus from [7]. We start Section three with a review of multiple Wiener-Itô integrals of Hilbert space valued functions. Integrals like this appear in the Wiener chaos expansion and in second quantisation [12]. In Theorem 3.5, we give an $L^2$-bound for the disentangled polynomials $T_{\mu,W;\nu}^{m,n}(A_1, A_2)$, for $m, n = 1, 2, \ldots$, where $x$ is an element of the underlying Hilbert space $H$ and $\mu$ is a continuous Borel measure. The bound depends on the Itô isometry and so on the underlying Hilbert space structure. The observation that for each $x \in X$, the $H$-valued process $t \mapsto f_{\mu,W;\nu}(A_1, A_2)x$, $0 \leq t \leq T$, has a continuous modification is proved in Proposition 3.6.

The results are applied in Section four to the exponential function $f(z_1, z_2) = e^{z_1 + z_2}$, $z_1, z_2 \in \mathbb{C}$, so that the solution the the stochastic equation (1.2) is represented in the linear space $\mathcal{L}(H, L^2(\mathbb{P}, H))$ as $X_t = f_{\lambda,W;\nu}(A, B)$, $0 \leq t \leq T$. Furthermore, $f_{\lambda,W;\nu}(A, B)$ has a representation as a stochastic Dyson series (1.4). We end Section four with some observations about the possibility of extending the results to Banach space valued processes.
5, we give some results for unbounded operators \( A, B \) defined in Hilbert space which suggest that the stochastic functional calculus \( f \mapsto f_{\mathcal{H},W}(A,B) \) may be defined whenever \( A \) has a sufficiently rich \((\mathcal{H}^*)\) functional calculus and \( B \) is a perturbation of \( A \), like a first order differential operator in the case that \( A \) is an elliptic differential operator.

2 Feynman’s Operational Calculus

Let \( E \) be a Banach space and let \( A_1, \ldots, A_n \) be nonzero bounded linear operators \( E \). We first introduce a commutative Banach algebra consisting of ‘analytic functions’ \( f(\tilde{A}_1, \ldots, \tilde{A}_n) \), where \( \tilde{A}_1, \ldots, \tilde{A}_n \) are treated as purely formal commuting objects. The collection \( \mathbb{D} = \mathbb{D}(A_1, \ldots, A_n) \) consists of all expressions of the form

\[
f(\tilde{A}_1, \ldots, \tilde{A}_n) = \sum_{m_1, \ldots, m_n = 0}^{\infty} c_{m_1, \ldots, m_n} \tilde{A}_1^{m_1} \cdots \tilde{A}_n^{m_n} \quad (2.1)
\]

where \( c_{m_1, \ldots, m_n} \in \mathbb{C} \) for all \( m_1, \ldots, m_n = 0, 1, \ldots \), and

\[
\|f(\tilde{A}_1, \ldots, \tilde{A}_n)\| = \|f(\tilde{A}_1, \ldots, \tilde{A}_n)\|_{\mathbb{D}(A_1, \ldots, A_n)} := \sum_{m_1, \ldots, m_n = 0}^{\infty} |c_{m_1, \ldots, m_n}| \|A_1\|^{m_1} \cdots \|A_n\|^{m_n} < \infty. \quad (2.2)
\]

The norm on \( \mathbb{D}(A_1, \ldots, A_n) \) defined by \( (2.2) \) makes \( \mathbb{D}(A_1, \ldots, A_n) \) into a commutative Banach algebra under pointwise operations. We refer to \( \mathbb{D}(A_1, \ldots, A_n) \) as the disentangling algebra associated with the \( n \)-tuple \( (A_1, \ldots, A_n) \) of bounded linear operators acting on \( E \). We can identify elements of \( \mathbb{D}(A_1, \ldots, A_n) \) with absolutely convergent power series in a polydisk but it is more suggestive to consider \( \tilde{A}_1, \ldots, \tilde{A}_n \) as commuting avatars of the bounded linear operators \( A_1, \ldots, A_n \).

Fix \( t > s \geq 0 \). Let \( A_1, \ldots, A_n \) be nonzero operators from \( \mathcal{L}(E) \) and let \( \mu_1, \ldots, \mu_n \) be continuous measures defined at least on \( \mathcal{B}([s,t]) \), the Borel class of \([s,t] \). The total mass of a measure \( \mu \) is written as \( \|\mu\|_{[s,t]} \).

The idea is to replace the operators \( A_1, \ldots, A_n \) with the elements \( \tilde{A}_1, \ldots, \tilde{A}_n \) from \( \mathbb{D} = \mathbb{D}(\|\mu_1\|A_1, \ldots, \|\mu_n\|A_n) \) and then form the desired function of \( \tilde{A}_1, \ldots, \tilde{A}_n \). Still working in \( \mathbb{D} \), we time order the expression for the function and then pass back to \( \mathcal{L}(E) \) simply by removing the tildes.

Given nonnegative integers \( m_1, \ldots, m_n \), we let \( m = m_1 + \cdots + m_n \) and

\[
P^{m_1, \ldots, m_n}(z_1, \ldots, z_n) = z_1^{m_1} \cdots z_n^{m_n}, \quad \text{for } z_1, \ldots, z_n \in \mathbb{C}. \quad (2.3)
\]

Then \( P^{m_1, \ldots, m_n}(\tilde{A}_1, \ldots, \tilde{A}_n) = \tilde{A}_1^{m_1} \cdots \tilde{A}_n^{m_n} \) as an element of \( \mathbb{D} \).

We are now ready to define the disentangling map \( T_{\mu_1, \ldots, \mu_n} \), which will return us from our commutative framework \( \mathbb{D}(A_1, \ldots, A_n) \) to the noncommutative setting of \( \mathcal{L}(E) \). For \( i = 1, \ldots, m \), we define

\[
C_i := \begin{cases} A_1 & \text{if } i \in \{1, \ldots, m_1\}, \\ A_2 & \text{if } i \in \{m_1 + 1, \ldots, m_1 + m_2\}, \\ \vdots & \vdots \\ A_n & \text{if } i \in \{m_1 + \cdots + m_{n-1} + 1, \ldots, m\}. \end{cases} \quad (2.4)
\]
For each } m = 0, 1, \ldots , \text{ let } S_m \text{ denote the set of all permutations of the integers } \{1, \ldots , m\}, \text{ and given } \pi \in S_m, \text{ we let }

\Delta_m(\pi; s, t) = \{(s_1, \ldots , s_m) \in [s, t]^m : s < s_{\pi(1)} < \cdots < s_{\pi(m)} < t\}.

If } \pi \text{ is the identity, then we write } \Delta_m(s, t) \text{ instead. We write } \Delta_m(\pi; t) \text{ and } \Delta_m(t) \text{ if } s = 0.

**Definition 2.1.** \( T_{m_1, \ldots , m_n} (p_{m_1, \ldots , m_n} (\tilde{A}_1, \ldots , \tilde{A}_n)) := \)
\[
\sum_{\pi \in S_m} \int_{\Delta_m(\pi; s, t)} C_{\pi(m)} \cdots C_{\pi(1)} (\mu_1^{m_1} \times \cdots \times \mu_n^{m_n}) (ds_1, \ldots , ds_m). \tag{2.5}
\]

The notation } \mu_j^0 \text{ means that the integral with respect to the } s_j\text{-variable is simply omitted. We adopt this convention even if } \mu_j \text{ is the zero measure.}

Then, for } f(\tilde{A}_1, \ldots , \tilde{A}_n) \in \mathcal{D}(A_1, \ldots , A_n) \text{ given by }
\[
f(\tilde{A}_1, \ldots , \tilde{A}_n) = \sum_{m_1, \ldots , m_n = 0}^{\infty} c_{m_1, \ldots , m_n} \tilde{A}_1^{m_1} \cdots \tilde{A}_n^{m_n}, \tag{2.6}
\]
we set \( T_{m_1, \ldots , m_n} (f(\tilde{A}_1, \ldots , \tilde{A}_n)) \) equal to
\[
\sum_{m_1, \ldots , m_n = 0}^{\infty} c_{m_1, \ldots , m_n} T_{m_1, \ldots , m_n} (p_{m_1, \ldots , m_n} (\tilde{A}_1, \ldots , \tilde{A}_n)). \tag{2.7}
\]

In the commutative setting and with probability measures, the right-hand side of (2.5) gives us what we would expect [7, Proposition 2.2], namely } p_{m_1, \ldots , m_n} (\tilde{A}_1, \ldots , \tilde{A}_n).

We shall sometimes write the bounded linear operator \( T_{m_1, \ldots , m_n; s, t} (f(\tilde{A}_1, \ldots , \tilde{A}_n)) \) as } f_{m_1, \ldots , m_n; s, t} (A_1, \ldots , A_n). \text{ In particular, }
\[
p_{m_1, \ldots , m_n} (A_1, \ldots , A_n) = T_{m_1, \ldots , m_n; s, t} (p_{m_1, \ldots , m_n} (\tilde{A}_1, \ldots , \tilde{A}_n)). \tag{2.8}
\]

The following result appeared in [8, Corollary 5.3] in the case that } t = 1 \text{ and } s = 0. \text{ A similar proof works for the case below.}

**Theorem 2.2.** Let } E \text{ be a banach space and let } \mu \text{ and } \nu \text{ be continuous measures on the Borel } \sigma\text{-algebra of } [0, \infty]. \text{ Let } A, B \text{ be elements of } L(E). \text{ Then for all } t > s \geq 0,
\[
e_{m_1, \ldots , m_n}^{A+B} := T_{m_1, \ldots , m_n} (e^{A+B}) = e^{A(t,s)} + \sum_{n=1}^{\infty} \left[ \int_s^{t} \int_s^{t} \cdots \int_s^{t} e^{A(s_1, s_2)} Be^{A(s_1, s_2, \ldots , s_n)} ds_1 ds_2 \right]. \tag{2.9}
\]
Brownian time-ordering

It follows that $e^{A+B}_{\mu,\nu,t}$ satisfies the integral equation

$$e^{A+B}_{\mu,\nu,t} = e^{A\mu([s,t])} + \int_s^t e^{A\mu([r,t])} B e^{A+B}_{\mu,\nu,s,r} \, d\nu(r)$$

(2.10)

by substituting equation (2.9) into the right-hand side of equation (2.10). Feynman’s disentangling ideas suggest that for every $0 \leq r < s \leq t$, the equation

$$e^{A+B}_{\mu,\nu,t} e^{A+B}_{\mu,\nu,t} = e^{A+B}_{\mu,\nu,t}$$

(2.11)

ought to be valid, that is, $e^{A+B}_{\mu,\nu,t}$, $0 \leq s \leq t$, is an evolution system, see [14, Theorem 5.3.1]. A proof of equation (2.11) and a more general disentangling formula will appear in a forthcoming paper of the author and Jerry Johnson [10].

Formulae (2.9)-(2.11) above remain valid if $A$ is the generator of a $C_0$-semigroup $e^{A}$, $t \geq 0$, on $X$, if for each $0 \leq s < t$ we let

$$e^{A+B}_{\mu,\nu,t} = \lim_{k \to \infty} e^{A_k+B}_{\mu,\nu,t}$$

in the strong operator topology with the Yosida approximations $A_k = kA(kI-A)^{-1}$, $k = 1, 2, \ldots$, to $A$.

3 Stochastic Disentangling in Hilbert Space

Suppose that in the situation of the preceding section, $\mu(dt)$ is Lebesgue measure $dt$ and $\nu(dt)$ is integration with respect to “white noise” $dW_t$—an idea suggested by B. Goldys. Then the multiple integrals in the perturbation series expansion (2.9) need to be replaced by multiple stochastic integrals with respect to the Brownian motion process.

More precisely, let $W$ denote Brownian motion in $\mathbb{R}$ with respect to the probability measure space $(\Omega, S, \mathbb{P})$ such that $W_0 = 0$ almost surely. For convenience, $\Omega$ is taken to be the set of all continuous functions $\omega : [0, \infty) \to \mathbb{R}$, the $\sigma$-algebra $S$ is the Borel $\sigma$-algebra of $\Omega$ for the compact-open topology and $W_t(\omega) = \omega(t)$ for every $\omega \in \Omega$ and $t \geq 0$. Then Wiener measure $\mathbb{P}$ has the property that $W_t$, $t \geq 0$, is a process with independent increments such that $W_t$ is a gaussian random variable with mean zero and variance $t$ for $t > 0$.

For a Banach space $E$ and $1 \leq p < \infty$, the space of $E$-valued $p$th-Bochner integrable functions with respect to $\mathbb{P}$ is denoted by $L^p(\mathbb{P}, E) = L^p(\Omega, S, \mathbb{P}, E)$. The linear space $L^0(\mathbb{P}, E) = L^0(\Omega, S, \mathbb{P}, E)$ of strongly measurable $E$-valued functions has the (metrisable) topology of convergence in probability.

3.1 Multiple stochastic integrals

For the purpose of expanding solutions of linear stochastic equations like (1.2) as a stochastic Dyson series (1.4), we need to consider multiple Wiener-Itô integrals of deterministic functions. We follow the account in [12, Section 10.3] with suitable modifications for vector valued functions.

Let $H$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$. Let $T > 0$ and $k = 1, 2, \ldots$. The case $k = 1$ corresponds to the Wiener integral. Let $D_1 = (0, T]$ and

$$D_k = \{(t_1, \ldots, t_k) \in (0, T]^k : \exists i, j = 1, \ldots, k, i \neq j, \text{ such that } t_i = t_j\}, \quad k = 2, 3, \ldots$$
Let $A_1, \ldots, A_n$ be a partition of $[0, T]$ into disjoint intervals of the form $(s, t]$ for $0 \leq s < t \leq T$ and suppose that
\[
f(t_1, \ldots, t_k) = \sum_{1 \leq j_1, \ldots, j_k \leq n} \alpha_{j_1, \ldots, j_k} \chi_{A_{j_1} \times \cdots \times A_{j_k}}
\] (3.1)
is a $H$-valued function such that $\alpha_{j_1, \ldots, j_k} = 0$ whenever two indices $j_1, \ldots, j_k$ are equal and $f$ vanishes on $D_k$. Then
\[
I_k(f) = \int_{[0, T]^k} f(t_1, \ldots, t_k) \, dW_{t_1} \cdots dW_{t_k}
\]
is defined by
\[
I_k(f) = \sum_{1 \leq j_1, \ldots, j_k \leq n} \alpha_{j_1, \ldots, j_k} W(A_{j_1}) \cdots W(A_{j_k})
\]
Here $W((s, t])$ denotes the random variable $W_t - W_s$ for $0 \leq s < t \leq T$. Let $D((0, T]^k, H)$ denote the linear space of $H$-valued step functions $f$ of the above form. Then $I_k$ is well-defined and $I_k : D((0, T]^k, H) \to L^2(\Omega, S, \mathbb{P}, H)$ is a linear map. Moreover, the maps $I_k$, $k = 1, 2, \ldots$, enjoy the following properties.

1) The integral $I_k(f)$ is invariant under the symmetrisation of the function $f \in D((0, T]^k, H)$, that is, if $\tilde{f} \in D((0, T]^k, H)$ is the symmetrisation
\[
\tilde{f}(t_1, \ldots, t_k) = \frac{1}{k!} \sum_{\sigma \in S_k} f(t_{\sigma(1)}, \ldots, t_{\sigma(k)}), \quad t_1, \ldots, t_k \in (0, T]
\]
of $f$, then $I_k(f) = I_k(\tilde{f})$.

2) If $k$ and $k'$ are positive integers such that $k \neq k'$ and $f \in D((0, T]^k, H)$, $g \in D((0, T]^{k'}, H)$, then $\mathbb{E}(I_k(f), I_{k'}(g)) = 0$.

3) If $f \in D((0, T]^k, H)$ and $g \in D((0, T]^k, H)$, then
\[
\mathbb{E}(I_k(f), I_k(g)) = k! \langle \tilde{f}, \tilde{g} \rangle_{L^2((0, T]^k, H)}.
\]
The inner product on the right hand side is taken in the Hilbert space $L^2((0, T]^k, H)$.

By property 3), we have a version of the Itô isometry
\[
\mathbb{E}(\|I_k(f)\|_H^2) = \mathbb{E}(\|I_k(\tilde{f})\|_H^2) = k! \|\tilde{f}\|_{L^2((0, T]^k, H)}^2 \leq k! \|f\|_{L^2((0, T]^k, H)}^2,
\] (3.2)
so that the mapping $I_k$ can be extended to a bounded linear operator
\[
I_k : L^2((0, T]^k) \to L^2(\Omega, S, \mathbb{P}, H).
\]
We also write $I_k(f)$ as $\int_{[0, T]^k} f(s) W^k(ds)$. In the case that $0 \leq s < t \leq T$ and $f \in L^2((0, T]^k, H)$ is zero off $\Delta_k(s, t)$, then
\[
I_k(f) = \int_s^t \cdots \int_s^t f(t_1, \ldots, t_k) \, dW_{t_1} \cdots dW_{t_k},
\] (3.3)
where the right-hand side is interpreted as an iterated stochastic integral. The equality is easily seen to be valid for all \( f \in \mathcal{D}((0,T]^k, H) \) vanishing off \( \Delta_k(s,t) \) and the linear subspace of all such functions is dense in the closed subspace of \( L^2((0,T]^k, H) \) consisting of all \( H \)-valued functions belonging to \( L^2((0,T]^k, H) \) which are zero almost everywhere outside \( \Delta_k(s,t) \subset (0,T]^k \).

Using Itô’s formula to compute \( \int_{\Delta_k(t)} W^k(ds_1, \ldots, ds_k) \) for \( k = 1, 2, \ldots \), we have

\[
\int_0^t W(ds_1) = W_t, \quad [k = 1]
\]

\[
\int_0^t \int_0^{s_2} W(ds_1)W(ds_2) = \int_0^t W_s W(ds_2) = \frac{1}{2} W_t^2 - \frac{1}{2} t, \quad [k = 2]
\]

\[
\int_0^t \int_0^{s_3} \int_0^{s_2} W(ds_1)W(ds_2)W(ds_3) = \int_0^t \left( \frac{1}{2} W_{s_3}^2 - \frac{1}{2} s_3 \right) W(ds_3) = \frac{1}{3!} W_t^3 - \frac{1}{2} t W_t, \quad [k = 3]
\]

\[
\vdots
\]

\[
\int_{\Delta_k(t)} W^k(ds_1, \ldots, ds_k) = \frac{1}{k!} h_k(W_t/\sqrt{t}) t^{k/2},
\]

where \( h_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}, \ x \in \mathbb{R} \), is Hermite polynomial of degree \( n = 0, 1, 2, \ldots \), see [12, Theorem 10.3.2].

Note that by symmetry, we have

\[
\int_{\Delta_k(\pi,t)} W^k(ds_1, \ldots, ds_k) = \int_{\Delta_k(t)} W^k(ds_1, \ldots, ds_k)
\]

for each \( \pi \in S_k \), so by equation (3.2) the equality

\[
\left\| \int_{\Delta_k(t)} W^k(ds_1, \ldots, ds_k) \right\|_{L^2(P)}^2 = \frac{t^k}{k!}
\]

holds for each \( k = 1, 2, \ldots \). This may also be obtained by applying the Itô isometry consecutively to the representation (3.3) or observing that

\[
\left\| h_k(W_t/\sqrt{t}) \right\|_{L^2(P)}^2 = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h_k(x)^2 e^{-x^2/2} dx = k!.
\]

**Lemma 3.1.** Let \( H \) be a Hilbert space, \( t > 0, k = 1, 2, \ldots \) and \( f : [0,t] \to L(H) \) an operator valued function such that \( f(\cdot):x \) is strongly Borel measurable in \( H \) and \( \|f(\cdot)\| \) is essentially bounded on \([0,t]\). If \( g \in L^2(\Delta_k(t), H) \), then the function

\[
f g : (s_1, \ldots, s_{k+1}) \mapsto f(s_{k+1})g(s_1, \ldots, s_k), \quad (s_1, \ldots, s_{k+1}) \in \Delta_{k+1}(t)
\]

belongs to \( L^2(\Delta_{k+1}(t), H) \) and the equality

\[
\int_{\Delta_{k+1}(t)} fg dW^{k+1} = \int_0^t f(s_{k+1}) \left( \int_{\Delta_k(s_{k+1})} g dW^{k+1} \right) dW(s_{k+1}) \quad (3.4)
\]
Then above. Appealing to Lemma 3.2 and the bound (3.2), we note that
\[ \int_{\Delta_k(t)} fg \, dW^{k+1} = \int_{\Delta_k(t)} f(s_{k+1}) x \chi_B(s_1, \ldots, s_k) W^{k+1}(ds_1, \ldots, ds_{k+1}) \]
\[ = \int_0^t \int_0^{t_{k+1}} \cdots \int_0^{t_{k+1}} f(s_{k+1}) x \chi_B(s_1, \ldots, s_k) dW_{s_1} \cdots dW_{s_{k+1}} \]
\[ = \int_0^t f(s_{k+1}) \left( \int_0^{t_{k+1}} \cdots \int_0^{t_{k+1}} x \chi_B(s_1, \ldots, s_k) dW_{s_1} \cdots dW_{s_k} \right) dW_{s_{k+1}} \]
\[ = \int_0^t f(s_{k+1}) \left( \int_{\Delta_k(s_{k+1})} g \, dW^{k+1} \right) dW(s_{k+1}) \]  
[by equation (3.3)]

By linearity, the equality (3.4) holds for all \( H \)-valued Borel simple functions \( g \) and so for all \( g \in L^2(\Delta_k(t), H) \).

We note the following obvious estimate.

**Lemma 3.2.** Let \( \mu \) be a finite measure, \( A \subset [0, t]^{m+n} \) a Borel set and
\[ A(\xi) = \{(s_1, \ldots, s_m, \xi_1, \ldots, \xi_n) \in A \}, \xi \in \mathbb{R}^n. \]
Then \( \int_{\mathbb{R}^n} \mu^n(A(\xi))^2 \, d\xi \leq \|\mu\|^{2m}_{[0,t]^{m+n}}. \)

Let \( A \subset [0, t]^{m+n} \) be a measurable set. The random variable \( (\mu^m \times W^n)(A) \) is defined by
\[ (\mu^m \times W^n)(A) = \int_A (\mu^m \times W^n)(ds_1, \ldots, ds_{m+n}) = \int_{[0,t]^n} \mu^m(A(s)) W^n(ds), \]
where the integral with respect to \( W^n \) is the multiple Wiener-Itô integral of order \( n \) defined above. Appealing to Lemma 3.2 and the bound (3.2), we note that
\[ \| (\mu^m \times W^n)(A) \|_2 \leq \sqrt{n!} \left( \int_{\mathbb{R}^n} \mu^n(A(\xi))^2 \, d\xi \right)^{\frac{1}{2}} \leq \sqrt{n!} \|\mu\|^{m}_{[0,t]} t^{n/2}. \]

(3.5)

Let \( E \) be a Banach space and \( A_1, A_2 \in \mathcal{L}(E) \). As in equation (2.4), we define
\[ C_i := \begin{cases} A_1 & \text{if } i \in \{1, \ldots, m\}, \\ A_2 & \text{if } i \in \{m+1, \ldots, m+n\}. \end{cases} \]
for \( m, n = 1, 2, \ldots \).
Lemma 3.4. Let $\mu$ be a continuous Borel measure on $[0, \infty)$ and let $E$ be a Banach space and $A_1, A_2 \in \mathcal{L}(E)$. The $\mathcal{L}(E)$-valued random variable $\mathcal{T}_{\mu,W,t}(p^{m,n}(A_1, A_2))$ is defined for each $t > 0$ by

$$
\mathcal{T}_{\mu,W,t}(p^{m,n}(A_1, A_2)) := \sum_{\pi \in S_{m+n}} \int_{\Delta_{m+n}(\pi;0,t)} C_{\pi(m+n)} \cdots C_{\pi(1)}(\mu^m \times W^n)(ds_1, \ldots, ds_{m+n}).
$$

(3.6)

The notation $\mu^0$ or $W^0$ means that the corresponding integral is simply omitted, with the understanding that $\mathcal{T}_{\mu,W,t}(p^{0,0}(A_1, A_2)) = I \ \mathbb{P}$-a.e.

Note that this expression is just a finite sum of operators times real valued random variables. For each $x \in E$, we take $\mathcal{T}_{\mu,W,t}(p^{m,n}(A_1, A_2)) x$ to be the $E$-valued random variable

$$
\sum_{\pi \in S_{m+n}} \int_{\Delta_{m+n}(\pi;0,t)} (C_{\pi(m+n)} \cdots C_{\pi(1)} x)(\mu^m \times W^n)(ds_1, \ldots, ds_{m+n}).
$$

Also, we have

$$
\mathcal{T}_{\mu,W,t} p^{m,n}(A_1, A_2)
$$

$$
= m! n! \sum_{\pi \in S_{m+n}} \int_{(\Delta_{m} \times \Delta_n)(\pi;0,t)} C_{\pi(m+n)} \cdots C_{\pi(1)}(\mu^m \times W^n)(ds_1, \ldots, ds_{m+n})
$$

(3.7)

where

$$
\Delta_m(t) := \{(s_1, \ldots, s_m) \in [0,t]^m : 0 < s_1 < \cdots < s_m < t\},
$$

$$
\Delta_n(t) := \{(s_{m+1}, \ldots, s_{m+n}) \in [0,t]^n : 0 < s_{m+1} < \cdots < s_{m+n} < t\},
$$

(3.8)

and

$$
(\Delta_{m} \times \Delta_n)(\pi;0,t) := \{(s_1, \ldots, s_{m+n}) \in \Delta_m(t) \times \Delta_n(t) : 0 < s_{\pi(1)} < \cdots < s_{\pi(m)} < t\},
$$

(3.9)

and where $P_{m,n}$ is the set of permutations $\pi \in S_n$ which leave $s_1, \ldots, s_m$ in the same relative order and $s_{m+1}, \ldots, s_{m+n}$ in the same relative order, that is, $\pi^{-1}(j) < \pi^{-1}(j+1)$ for $j = 1, \ldots, m-1$ and $j = m+1, \ldots, m+n-1$. The cardinality of $P_{m,n}$ is given by

$$\text{Card}(P_{m,n}) = \frac{(m+n)!}{m!n!},
$$

(3.10)

because for each of the $(m+n)!$ permutations $\pi$ of the set $\{1, \ldots, m+n\}$, there exists a unique permutation $\alpha_\pi \in P_{m,n}$ for which $\pi^{-1}(1, \ldots, m) = \alpha_\pi^{-1}(1, \ldots, m)$ and $\pi^{-1}(m+1, \ldots, m+n) = \alpha_\pi^{-1}(m+1, \ldots, m+n)$ and unique bijections $j_1 : \{1, \ldots, m\} \to \{1, \ldots, m\}$ and $j_2 : \{m+1, \ldots, m+n\} \to \{m+1, \ldots, m+n\}$ such that $\pi = j_1 \circ \alpha_\pi$ on $\pi^{-1}(1, \ldots, m)$ and $\pi = j_2 \circ \alpha_\pi$ on $\pi^{-1}(m+1, \ldots, m+n)$.

Lemma 3.4. Let $E$ be a Banach space and $A_1, A_2 \in \mathcal{L}(E)$. Then

$$
\mathcal{T}_{\mu,W,t} p^{m,n}(A_1, A_2)
$$

$$
= \sum_{j_0 + \cdots + j_n = m} \frac{m! n!}{j_0! \cdots j_n!} A_1^{j_0} A_2 A_1^{j_1} \cdots A_1^{j_n} A_2 A_1^{j_0}
$$

$$
\times \int_{\Delta_n(t)} \mu([0,s_1])^{j_0} \mu([s_1,s_2])^{j_1} \cdots \mu([s_n,t])^{j_n} W^n(ds_1, \ldots, ds_n)
$$

(3.11)
Proof. Suppose in the equality (3.7) we take a fixed $\pi \in \mathcal{P}_{m,n}$. Then $C_{\pi(j)} = B$ for all $j \in \pi^{-1}\{m + 1, \ldots, m + n\}$ and $C_{\pi(j)} = A$ for all $j \in \pi^{-1}\{1, \ldots, m\}, j = 1, \ldots, m + n$. For each $k = 1, \ldots, n - 1$, let $j_k$ be the number of integers greater than $\pi^{-1}(m + k)$ and less than $\pi^{-1}(m + k + 1)$ and let $j_0 = \pi^{-1}(m + k) - 1, j_n = (m + n) - \pi^{-1}(m + n)$ and we have

$$\int_{(\Delta_n \times \Delta_n)(\pi, t)} C_{\pi(m+n)} \cdots C_{\pi(1)}(\mu^n \times W^n)(ds_1, \ldots, ds_{m+n})$$

$$= A_{j_0}^{j_0} A_{2A_1}^{j_1} \cdots A_{1}^{j_1} A_{2A_1}^{j_0} (\mu^n \times W^n)((\Delta_m \times \Delta_n)(\pi, t))$$

$$= \frac{1}{j_0! \cdots j_n!} A_{j_0}^{j_0} A_{2A_1}^{j_1} \cdots A_{1}^{j_1} A_{2A_1}^{j_0} \times \int_{\Delta_n(t)} \mu(0, s_1)A_{j_0}^{j_0} \mu([s_1, s_2])A_{j_1}^{j_1} \cdots \mu([s_n, t])A_{j_n}^{j_n} W^n (ds_1, \ldots, ds_n)$$

because each section $((\Delta_m \times \Delta_n)(\pi, t))(s_{m+1}, \ldots, s_{m+n})$ is a product set

$$\{0 < s_{\pi(1)} < \cdots < s_{\pi(j_0)} < s_{m+1} \} \times \cdots \times \{s_m < s_{\pi(m-j_0+1)} < \cdots < s_{\pi(n)} < t\}$$

and each set $\{a < s_1 < \cdots < s_k < b\}$ has $\mu$-measure $\mu([a, b])^k/k!$ because $\mu$ is continuous.

Each choice of nonnegative integers $j_0, \ldots, j_n$ such that $j_0 + \cdots + j_n = m$ uniquely determines the permutation $\pi \in \mathcal{P}_{m,n}$, so equation (3.11) follows from equation (3.7). \qed

**Theorem 3.5.** Let $H$ be a Hilbert space and $A_1, A_2 \in \mathcal{L}(H)$. Then for each $x \in H$ and $m, n = 0, 1, \ldots$, we have

$$\|T_{\mu,W}^n P^n(A_1, A_2)x\|_{L^2(\mathcal{P}, H)} \leq \sqrt{n!} (\|\mu\|_{[0,1]} \|A_1\| \|A_2\|)^{m} (1 + \|A_2\|^2)^n \|x\|. \quad (3.12)$$

Proof. Let $x \in H$. First we note that by Lemma 3.4,

$$(T_{\mu,W}^n P^n(A_1, A_2)) x$$

$$= \sum_{\pi \in \mathcal{P}_{m,n}} \int_{\Delta_n(\pi, t)} C_{\pi(m+n)} \cdots C_{\pi(1)}(\mu^n \times W^n)(ds_1, \ldots, ds_{m+n})$$

$$= \sum_{j_0 + \cdots + j_n = m} \frac{m!}{j_0! \cdots j_n!} A_{j_0}^{j_0} A_{2A_1}^{j_1} \cdots A_{1}^{j_1} A_{2A_1}^{j_0} x$$

$$\times \mu([0, s_1])A_{j_0}^{j_0} \mu([s_1, s_2])A_{j_1}^{j_1} \cdots \mu([s_n, t])A_{j_n}^{j_n} W^n (ds_1, \ldots, ds_n)$$

Applying the Itô isometry (3.2), we have

$$\| (T_{\mu,W}^n P^n(A_1, A_2)) x \|^2_{L^2(\mathcal{P}, H)}$$

$$= n! \int_{[0,1]^n} \left\| \sum_{\pi \in \mathcal{P}_n} \chi_{\Delta_n(\pi, t)}(s_1, \ldots, s_n) \right\|^2 \sum_{j_0 + \cdots + j_n = m} \frac{m!}{j_0! \cdots j_n!} A_{j_0}^{j_0} A_{2A_1}^{j_1} \cdots A_{1}^{j_1} A_{2A_1}^{j_0} x$$

$$\times \mu([0, s_{\pi(1)}])A_{j_0}^{j_0} \mu([s_{\pi(1)}, s_{\pi(2)}])A_{j_1}^{j_1} \cdots \mu([s_{\pi(n)}, t])A_{j_n}^{j_n} W^n (ds_1, \ldots, ds_n)$$

$$\leq \sqrt{n!} (\|\mu\|_{[0,1]} \|A_1\| \|A_2\|)^{m} (1 + \|A_2\|^2)^n \|x\||x||.$$

\(\square\)
\[
\begin{align*}
&= n^2 \int_{\Delta(t)} \left\| \sum_{j_0 + \cdots + j_n = m} \frac{m!}{j_0! \cdots j_n!} A_{j_0}^1 A_{j_1}^n \cdots A_{j_n}^1 A_{2}^0 \right\|_H^2 \, ds_1 \ldots ds_n \\
&\times \mu([0,s_1])^{j_0} \mu([s_1,s_2])^{j_1} \cdots \mu([s_{n-1},t])^{j_{n-1}} \mu([s_n,t])^{j_n} \, ds_1 \ldots ds_n \\
&\leq n^2 \|A_1\|^{2m} \|A_2\|^{2n} \|x\|^2 \int_{\Delta(t)} \left( \sum_{j_0 + \cdots + j_n = m} \frac{m!}{j_0! \cdots j_n!} \right)^2 \, ds_1 \ldots ds_n \\
&\times \mu([0,s_1])^{j_0} \mu([s_1,s_2])^{j_1} \cdots \mu([s_{n-1},t])^{j_{n-1}} \mu([s_n,t])^{j_n} \, ds_1 \ldots ds_n \\
&= n^2 \|A_1\|^{2m} \|A_2\|^{2n} \|x\|^2 \int_{\Delta(t)} \mu([0,r])^{2m} ds_1 \ldots ds_n \quad \text{[multinomial formula]} \\
&= n! \|A_1\|^{2m} \|A_2\|^{2n} \|x\|^2 \mu([0,r])^{2m}.
\end{align*}
\]
Hence \[\left\| \left( T_{\mu,W,t} P_{m,n} (\tilde{A}_1, \tilde{A}_2) \right) \right\|_{L(H, L^2(\mathbb{P}, H))} \leq \sqrt{n!} \left( \|\mu\|_{[0,r]} \|A_1\| \right)^m \left( \|\mu\|_{[0,r]} \|A_2\| \right)^n. \]

The collection \( D_W (A_1, A_2) \) consists of all expressions of the form
\[
\tilde{f}(\tilde{A}_1, \tilde{A}_2) = \sum_{m_1, m_2=0}^{\infty} c_{m_1, m_2} \tilde{A}_1^{m_1} \tilde{A}_2^{m_2} 
\]
where \( c_{m_1, m_2} \in \mathbb{C} \) for all \( m_1, m_2 = 0, 1, \ldots, \) and
\[
\|\tilde{f}(\tilde{A}_1, \tilde{A}_2)\| = \|f(\tilde{A}_1, \tilde{A}_2)\|_{D_W (A_1, A_2)} \\
:= \sum_{m_1, m_2=0}^{\infty} \sqrt{m_2!} |c_{m_1, m_2}| \|A_1\|^{m_1} \|A_2\|^{m_2} < \infty. \tag{3.14}
\]

Then, for \( f(\tilde{A}_1, \tilde{A}_2) \in D_W ([\mu]_{[0,t]} A_1, t^{\frac{1}{2}} A_2) \) given by
\[
f(\tilde{A}_1, \tilde{A}_2) = \sum_{m_1, m_2=0}^{\infty} c_{m_1, m_2} \tilde{A}_1^{m_1} \tilde{A}_2^{m_2}, \tag{3.15}
\]
we set \( f_{\mu,W,t} (A_1, A_2) := T_{\mu,W,t} (f(\tilde{A}_1, \tilde{A}_2)) \) equal to
\[
\sum_{m_1, m_2=0}^{\infty} c_{m_1, m_2} \left( T_{\mu,W,t} (P_{m_1,m_2} (\tilde{A}_1, \tilde{A}_2)) \right). \tag{3.16}
\]

According to Theorem 3.5, the series converges absolutely in the strong operator topology of the space \( L(H, L^2(\mathbb{P}, H)) \) of random linear operators [15].

**Proposition 3.6.** Let \( H \) be a Hilbert space and \( A_1, A_2 \in L(H) \). Suppose that \( T > 0 \) and \( \mu \) is a continuous measure on the Borel \( \sigma \)-algebra of \([0, T]\). Let \( f(\tilde{A}_1, \tilde{A}_2) \in D_W ([\mu]_{[0,t]} A_1, t^{\frac{1}{2}} A_2) \).

Then \( t \mapsto f_{\mu,W,t} (A_1, A_2) x, \ 0 \leq t \leq T, \) is a continuous function with values in \( L^2(\mathbb{P}, H) \) for each \( x \in H \).

Furthermore, for each \( x \in H \), the vector valued process \( \langle f_{\mu,W,t} (A_1, A_2) x \rangle_{0 \leq t \leq T} \) has a pathwise continuous modification — there exists a strongly progressively measurable function \( \Phi : [0, T] \times \Omega \to H \), such that \( \Phi(t, \omega) = f_{\mu,W,t} (A_1, A_2) x \) (\( \mathbb{P} \) a.e.) for each \( 0 \leq t \leq T \) and for each \( \omega \in \Omega \), the function \( t \mapsto \Phi(t, \omega), \ 0 \leq t \leq T, \) is norm continuous in \( H \).
Proof. Let \( x \in H \). For each \( 0 \leq t \leq T \), the bound
\[
\| T_{\mu,W,t} \left( p^{m_1,m_2}(\tilde{A}_1,\tilde{A}_2) \right)x \|_{L^2(\mathbb{P},H)} \leq \sqrt{n} \left( \| \mu_{[0,t]} \|_{0,T} \right)^n (T^{1/2} \| A_2 \|)^n \| x \| \quad (3.17)
\]
holds for every \( m,n = 0,1,\ldots \). Because \( T_{\mu,W,t} \left( p^{m_1,m_2}(\tilde{A}_1,\tilde{A}_2) \right) \) is just the finite sum of bounded linear operators times multiple stochastic integrals of deterministic scalar functions, the function \( t \mapsto T_{\mu,W,t} \left( p^{m_1,m_2}(\tilde{A}_1,\tilde{A}_2) \right) \) is continuous from the interval \([0,T]\) into \( L^2(\mathbb{P},L(H)) \). By dominated convergence, \( t \mapsto f_{\mu,W,t}(A_1,A_2)x, \quad 0 \leq t \leq T, \) is a continuous function with values in \( L^2(\mathbb{P},H) \).

The existence of a continuous modification follows from a standard stopping time argument [13, Section 2.5]. For each \( k = 1,2,\ldots \), let \( M_k \) be a positive integer and let \( t \mapsto \Phi_k(t,\cdot) \) be a continuous modification of \( \sum_{m_1,m_2=0}^{M_k} c_{m_1,m_2} p^{m_1,m_2}_{\mu,W,t}(A_1,A_2)x \) such that
\[
\mathbb{E}( \| f_{\mu,W,t}(A_1,A_2)x - \Phi_k(t,\cdot) \|_H^2 ) \leq 8^{-k}.
\]
for all \( 0 \leq t \leq T \). Such a modification exists because \( T_{\mu,W,t} \left( p^{m_1,m_2}(\tilde{A}_1,\tilde{A}_2) \right) \) is just the finite sum of bounded linear operators times scalar multiple stochastic integrals.

Let \( \tau_n(\omega) = \inf \{ t : \| \Phi_n(t,\omega) - \Phi_{n+1}(t,\omega) \|_H > 2^{-n} \} \wedge T \). The bound (3.17) also holds if \( t \in [0,T] \) is replaced by the stopping time \( \tau_n \) by writing \( \tau_n \) as the decreasing limit of a sequence of simple stopping times, so
\[
2.8^{-n} \geq \mathbb{E}( \| \Phi_n(\tau_n,\cdot) - \Phi_{n+1}(\tau_n,\cdot) \|_H^2 ) \geq 4^{-n} \mathbb{P}( \tau_n^{-1}([0,T]))
\]
Let \( G = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \tau_n^{-1}([0,T]) \). Then \( \mathbb{P}(G) = 0 \) and for each \( \omega \in \Omega \setminus G \), the sequence \( \{ \Phi_n(t,\omega) \}_{n=1}^{\infty} \) converges uniformly for \( t \in [0,T] \). Because \( \{ \Phi_n(t,\cdot) \}_{n=1}^{\infty} \) also converges almost everywhere, it follows that off some null set of \( \omega \in \Omega \), the \( H \)-valued sequence \( \{ \Phi_n(t,\omega) \}_{n=1}^{\infty} \) converges uniformly for \( t \in [0,T] \) to a continuous function \( t \mapsto \Phi(t,\omega) \), \( t \in [0,T] \), such that \( \Phi \) is a modification of the \( H \)-valued process \( t \mapsto f_{\mu,W,t}(A_1,A_2)x \), \( 0 \leq t \leq T \).

4 Stochastic Disentangling an Exponential Factor

If we apply the stochastic disentangling procedure to the exponential function \( (z_1,z_2) \mapsto e^{z_1+z_2}, \quad z_1,z_2 \in \mathbb{C} \), then we obtain a representation of a linear stochastic differential equation in Hilbert space as a stochastic Dyson series (1.4) mentioned above.

Theorem 4.1. Let \( H \) be a Hilbert space and \( x \in H \). Let \( A \) and \( B \) be bounded linear operators on \( H \). Suppose that \( \mu : B([0,T]) \rightarrow [0,\infty) \) a continuous Borel measure, \( f(z_1,z_2) = e^{z_1+z_2} \) for all \( z_1,z_2 \in \mathbb{C} \) and \( Y_t = f_{\mu,W,t}(A,B)x, \quad t \in [0,T] \).

The \( H \)-valued random variable \( Y_t \) is given by the stochastic Dyson series
\[
Y_t = e^{\mu([0,t])} x + \sum_{n=1}^{\infty} \int_{\Delta_n(t)} e^{\mu([s_1,t])} A B \cdots e^{\mu([s_{n-1},t])} A B e^{\mu([0,s_1])} A x W^n(ds_1,\ldots,ds_n), \quad (4.1)
\]
which converges absolutely in \( L^2(\mathbb{P},H) \). Furthermore, the bounds
\[
\left\| \int_{\Delta_n(t)} e^{\mu([s_1,t])} A B \cdots e^{\mu([s_{n-1},t])} A B e^{\mu([0,s_1])} A x W^n(ds_1,\ldots,ds_n) \right\|_{L^2(\mathbb{P},H)}
\]
Brownian time-ordering

$$\|x\|_{L^2(\mathbb{P}, H)} \leq \|x\|_{\mathcal{D}(\mu)} \|\mathcal{A}\| \frac{(t^{1/2} \|B\|)^n}{\sqrt{n!}}, \quad n = 1, 2, \ldots, \quad (4.2)$$

and

$$\|Y_t\|_{L^2(\mathbb{P}, H)} \leq \|x\|_{\mathcal{D}(\mu)} \|\mathcal{A}\| \sum_{n=0}^{\infty} \frac{(t^{1/2} \|B\|)^n}{\sqrt{n!}} \quad (4.3)$$

hold for all $t \geq 0$.

**Proof.** Because $Y_t = f_{\mu, W,t}(A, B)x = \sum_{m,n=0}^{\infty} \frac{1}{m!n!} P_{\mu, W,t}^{m,n}(A, B)x$, the bound (4.3) follows by summing the inequality (3.12) over $m, n = 0, 1, \ldots$. The convergence is absolute in $L^2(\mathbb{P}, H)$.

According to equation (3.11), we have

$$Y_t = \sum_{m,n=0}^{\infty} \frac{1}{m!n!} P_{\mu, W,t}^{m,n}(A, B)x,$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} (\mu([0, t])A)^m x + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{j_0 + \cdots + j_n = m} \frac{1}{j_0! \cdots j_n!} A^{j_n} B A^{j_{n-1}} \cdots A^{j_1} B A^{j_0} x$$

$$\times \int_{\Delta_n(t)} \mu([0, s_1])^{j_0} \mu([s_1, s_2])^{j_1} \cdots \mu([s_{n-1}, t])^{j_{n-1}} \mu([s_n, t])^{j_n} W^n(ds_1, \ldots, ds_n),$$

$$= e^{\mu([0, t])A} x + \sum_{n=1}^{\infty} \left( \sum_{m=0}^{\infty} \sum_{j_0 + \cdots + j_n = m} \frac{1}{j_0! \cdots j_n!} A^{j_n} B A^{j_{n-1}} \cdots A^{j_1} B A^{j_0} x$$

$$\times \int_{\Delta_n(t)} \mu([0, s_1])^{j_0} \mu([s_1, s_2])^{j_1} \cdots \mu([s_{n-1}, t])^{j_{n-1}} \mu([s_n, t])^{j_n} W^n(ds_1, \ldots, ds_n) \right).$$

By dominated convergence, for each $n = 1, 2, \ldots$, the sum

$$\sum_{m=0}^{\infty} \sum_{j_0 + \cdots + j_n = m} \frac{1}{j_0! \cdots j_n!} (A^{j_n} B A^{j_{n-1}} \cdots A^{j_1} B A^{j_0} x) \mu([0, s_1])^{j_0} \mu([s_1, s_2])^{j_1} \cdots \mu([s_{n-1}, t])^{j_{n-1}} \mu([s_n, t])^{j_n}$$

converges absolutely in $L^2(\Delta_n(t), H)$ to $e^{\mu([0, s_1])A} B \cdots e^{\mu([s_1, s_2])A} B e^{\mu([0, s_1])A} x$. An appeal to the Itô isometry (3.2) shows that

$$\sum_{m=0}^{\infty} \sum_{j_0 + \cdots + j_n = m} \frac{1}{j_0! \cdots j_n!} A^{j_n} B A^{j_{n-1}} \cdots A^{j_1} B A^{j_0} x$$

$$\times \int_{\Delta_n(t)} \mu([0, s_1])^{j_0} \mu([s_1, s_2])^{j_1} \cdots \mu([s_{n-1}, t])^{j_{n-1}} \mu([s_n, t])^{j_n} W^n(ds_1, \ldots, ds_n)$$

$$= \int_{\Delta_n(t)} (e^{\mu([0, s_1])A} B \cdots e^{\mu([s_1, s_2])A} B e^{\mu([0, s_1])A} x) W^n(ds_1, \ldots, ds_n),$$

and the bound (4.2) holds, giving the representation (4.1). \qed

Let $A$ and $B$ be bounded linear operators acting on a Hilbert space $H$. We say that a continuous progressively measurable $H$-valued process $Y$ satisfies the stochastic equation

$$dY_t = AY_t \, dt + BY_t \, dW_t.$$
if $Y_t = Y_0 + \int_0^t AY_s \, ds + \int_0^t BY_s \, dW_s$ for $t \geq 0$. Stochastic integration for Hilbert space valued functions is treated in [13].

**Corollary 4.2.** Let $A$ and $B$ be bounded linear operators acting on a Hilbert space $H$. Let $f(z_1, z_2) = e^{z_1 + z_2}$, $z_1, z_2 \in \mathbb{C}$ and let $\lambda$ denote Lebesgue measure on $\mathbb{R}$.

For each $x \in H$, the $H$-valued process $Y_t = f_{\lambda, W}(A, B)x$, $t \geq 0$, represents the solution of the stochastic equation

$$dY_t = AY_t \, dt + BY_t \, dW_t, \quad Y_0 = x,$$

(4.4)

the map $t \mapsto Y_t$, $t \geq 0$ is continuous in $L^2(\mathbb{P}, H)$ and

$$Y_t = e^{tA}x + \sum_{k=1}^{\infty} \int_{\Delta_k(t)} e^{(t-s_k)A}B \cdots e^{(s_2-s_1)A}B e^{s_1A}x W^k (ds_1, \ldots, ds_k)$$

(4.5)

converges absolutely in $L^2(\mathbb{P}, H)$. Moreover, $e^{(t-s)A}BY_s$, $0 \leq s \leq t$ is stochastically integrable on $[0, t]$ and $Y$ is the unique solution of the stochastic integral equation

$$Y_t = e^{tA}x + \int_0^t e^{(t-s)A}BY_s \, dW_s$$

(4.6)

and satisfies the bound

$$\|Y_t\|_{L^2(\mathbb{P}, H)} \leq \|x\|e^{tA} \sum_{n=0}^{\infty} \left( \frac{t^{\frac{2}{n} \|B\|}}{\sqrt{n!}} \right)^n$$

for all $t > 0$.

**Proof.** Appealing to the representation (4.1), we need to show that (4.5) is a solution of the stochastic equations (4.4) and (4.6). Under our present assumptions, it is well known that a Hilbert space valued process $Y$ is a solution of the stochastic equation (4.4) (a strong solution) if and only if it is a solution of (4.6) (a mild solution), see [3, Chapter 6, pp. 150–156] and there exists a unique mild solution [3, Theorem 6.7]. Other than the uniqueness, which follows from the contraction mapping principle, we reproduce the argument that a mild solution is a strong solution in the present simpler context.

According to Lemma 3.1, we have

$$\int_0^t e^{(t-s)A}B \left( \int_{\Delta(t)} e^{(t-s_k)A}B \cdots e^{(s_2-s_1)A}B e^{s_1A}x W^k (ds_1, \ldots, ds_k) \right) \, dW_s$$

$$= \int_{\Delta_{k+1}(t)} e^{(t-s_{k+1})A}B \cdots e^{(s_2-s_1)A}B e^{s_1A}x W^{k+1} (ds_1, \ldots, ds_{k+1}).$$

The series (4.5) converges absolutely in $L^2(\mathbb{P}, H)$ due to the bounds (4.2). By direct substitution, the $H$-valued process $Y_t$ satisfies equation (4.6) on appealing to the Itô isometry. Let $\xi \in H$. An appeal to the stochastic Fubini theorem [3, Theorem 4.18] and the observation
that \( A \int_0^t e^{(s-r)A} ds = e^{(t-r)A} - I \) gives
\[
\int_0^t \left( \int_0^s e^{(s-r)A} BY_r dW_s, A^* \xi \right) ds = \int_0^t \int_0^t e^{(s-r)A} BY_r, A^* \xi \right) ds dW_r
= \int_0^t \int_0^t (e^{(t-r)A} - I) BY_r, \xi \right) dW_r
= \langle Y_t - e^{tA}x, \xi \rangle - \int_0^t \langle BY_s, \xi \rangle dW_s,
\]
from which we obtain the equality
\[
\int_0^t \langle A(Y_s - e^{sA}x), \xi \rangle ds = \langle Y_t - e^{tA}x, \xi \rangle - \int_0^t \langle BY_s, \xi \rangle dW_s, \quad \xi \in H.
\]
Finally, the identity \( \int_0^t A e^{sA} x ds = e^{tA} x - x \) shows that equation (4.4) holds.

### 4.1 Remarks on linear stochastic equation in Banach spaces

The key to the proof of Corollary 4.2 is the bound (3.12) for \( P_{m_1,m_2}(A_1,A_2)x \), whose proof depended on the Itô isometry for \( H \)-valued functions. A similar estimate holds for \( M \)-type 2 Banach spaces, such as \( L^p \)-spaces for \( p \geq 2 \) [1]. In order to treat stochastic equations like (4.4) in a general Banach space setting, one can adopt a “weak” or Pettis-type approach to the stochastic integration of vector-valued functions—an approach that has proved valuable in tackling stochastic convolution with respect to cylindrical Wiener process [16] and may also be used to good effect for semigroups of operators [5, 6]. The approach of [16] may prove useful in obtaining bounds for \( P_{m_1,m_2}(A_1,A_2)x \) in the general Banach space setting.

### 5 Unbounded Perturbations in Hilbert Space

Suppose that \( H \) is Hilbert space. We now consider the stochastic equation
\[
du_t = Au_t dt + Bu_t dW_t, \quad u_0 = x.
\]
for an \( H \)-valued process \( u_t, \ t \geq 0 \) for the case in which \( A : D(A) \to H \) is the generator of \( C_0 \)-semigroup. We also suppose that \((V, \| \cdot \|_V) \) is a separable Banach space, \( D(A) \subset V \subset H \) with continuous inclusions and \( B : V \to H \) is bounded. Suppose also that there exists \( c > 0 \) such that
\[
\int_0^T \| e^{tA} x \|_V^2 dt \leq c \| x \|_H^2 \tag{5.2}
\]
for all \( x \in D(A) \).

According to [4], for \( \| B \|_{L(V,H)} \) sufficiently small, the stochastic equation (5.1) has a “mild” solution
\[
u(t) = e^{tA} u_0 + \int_0^t e^{(t-s)A} Bu(s) dW_s,
\]
satisfying
The subscript $\mathcal{P}$ denotes measurability with respect to the predictable $\sigma$-algebra. In forthcoming joint work of the author with Z. Brzeźniak, more is obtained from Flandoli’s result.

**Theorem 5.1.** Suppose that $A$ satisfies the bound (5.2), $B$ satisfies the bound

$$
\lim_{t \to 0^+} \sup_{\|x\|_H \leq 1} \int_0^t \|Be^{sA}x\|_H^2 ds < 1, \quad (5.3)
$$

Then for each $T > 0$, equation (5.1) has a mild solution $u(\cdot, t) = e^{A+B}_{\lambda,W,T}x$, $0 \leq t \leq T$, where there exist $\delta > 0$ and $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$, with $t_j - t_{j-1} < \delta$ for $j = 1, \ldots, n + 1$, such that $e^{A+B}_{\lambda,W,T}x$ has the representation

$$
e^{A+B}_{\lambda,W,T+k}x = e^{A+B}_{\lambda,W,k}x + \sum_{k=1}^{n} \int_0^s \int_0^{s_k} \cdots \int_0^{s_2} \left[ e^{(s-s_k)A}Be^{(s_k-s_{k-1})A} \cdots Be^{s_1A}e^{A+B}_{\lambda,W,k}x \right] dW_{s_1} \cdots dW_{s_k} \quad a.e.
$$

for $0 \leq s < t_{k+1} - t_k$ and $x \in \mathcal{D}(A)$. The convergence is in $L^2(\mathbb{P}; H)$.

The notation is justified by the observation that if $A_n$ and $B_n$, $n = 1, 2, \ldots$, are bounded linear operators for which the bounds (5.1) and (5.2) hold uniformly in $n = 1, 2, \ldots$, and $A_n \to A$ in the sense of strong resolvent convergence and $B_n \to B$ on $\mathcal{D}(A)$, then for each $x \in H$ and $0 \leq t \leq T$, the limit

$$
\lim_{n \to \infty} e^{A_n+B_n}_{\lambda,W,T}x = e^{A+B}_{\lambda,W,T}x
$$

converges in $L^2(\mathbb{P}, H)$. For each $n = 1, 2, \ldots$, the $L(H)$-valued random variable $e^{A_n+B_n}_{\lambda,W,T}$ is given by formula (3.16) with $f(z_1, z_2) = e^{z_1+z_2}$, $A_1 = A_n$, $A_2 = B_n$ and $\mu$ equal to Lebesgue measure.

The bound (5.2) is very close to conditions encountered in harmonic analysis for the existence of functional calculi for operators. First we review the relevant definitions. For each $0 < \nu < \pi/2$, let

$$
S_{\nu}(\mathbb{C}) = \{ -z : z \in \mathbb{C}, \ |\arg z| \leq \nu \} \cup \{ 0 \}
$$

be the sector in the left half-plane of angle $2\nu$. Let $S_{\nu}^\circ(\mathbb{C})$ denote its interior.

Suppose that $T : \mathcal{D}(T) \to H$ is a densely defined linear operator acting in the Hilbert space $H$. If $0 \leq \omega < \pi/2$, then $T$ is said to be of type $\omega$, if $\sigma(T) \subset S_{\omega}(\mathbb{C})$ and for each $\nu > \omega$, there exists $C_{\nu} > 0$ such that

$$
\|(zI - T)^{-1}\| \leq C_{\nu}|z|^{-1}, \quad z \notin S_{\nu}(\mathbb{C}) \quad (5.4)
$$

Then the bounded linear operator $f(T)$ is defined by

$$
f(T) = \frac{1}{2\pi i} \int_C (\lambda I - T)^{-1} f(\lambda) d\lambda. \quad (5.5)
$$

for any holomorphic function $f$ with sufficient decay at zero and infinity in $S_{\nu}(\mathbb{C})$. The contour $C$ can be taken to be $\{ z \in \mathbb{C} : \Re(z) = \tan \theta \Re(z), \ \Re z \leq 0 \}$, with $\omega < \theta < \nu$. In
particular, $T$ is the generator of an analytic semigroup in the sector $S_{(\pi/2-v)}(\mathbb{C})$ with half-angle $\pi/2 - v$ in the right half-plane by the formula

$$e^{zT} = \frac{1}{2\pi i} \int_{C} (\lambda I - T)^{-1} e^{\lambda z} d\lambda, \quad z \in S_{(\pi/2-v)}(\mathbb{C}),$$

for a suitable contour $C$. Let $H^\infty(S_{v}^{-}(\mathbb{C}))$ denote the space of uniformly bounded holomorphic functions defined in the open sector $S_{v}^{-}(\mathbb{C})$.

The operator $T$ of type $\omega -$ is said to have a bounded $H^\infty$-functional calculus if for each $\omega < v < \pi/2$, there exists an algebra homomorphism $f \mapsto f(T)$ from $H^\infty(S_{v}^{-}(\mathbb{C}))$ to $L(H)$ agreeing with (5.5) and a positive number $C_v$ such that $\|f(T)\| \leq C_v \|f\|_\infty$ for all $f \in H^\infty(S_{v}^{-}(\mathbb{C}))$.

**Example 5.2.** Now suppose that $A$ is of type $\omega -$, $V = D(-A)^{1/2}$ and $A$ satisfies equation (5.2). Then [4] $A$ satisfies the “square function estimates”

$$\int_0^\infty \|e^{A t}x\|_V^2 dt \leq c\|x\|_H^2.$$

If $A^*$ satisfies (5.2) too, then $A$ has a bounded $H^\infty$-functional calculus [2]. The conditions that we have imposed in order to prove the existence of of the random linear operator

$$e^{A_{\lambda, \omega}^+ B} : H \rightarrow L^2(\mathbb{P}, H)$$

and the existence of solutions of the stochastic equation (5.1) are close to those which imply that $A$ has a rich functional calculus, so it is tempting to speculate that if $A$ has an $H^\infty$-functional calculus and $B$ is small compared to $A$, does a joint functional calculus $f \mapsto f_{\lambda, \omega}(A,B) \in L(H, L^2(\mathbb{P}, H))$ exist, at least for small times $t$? If so, the solution of the stochastic equation (5.1) is represented simply by substituting the exponential function $(z_1, z_2) \mapsto e^{z_1 + z_2}$ for $f$ in the joint functional calculus with Brownian time-ordering. Such a representation would implement Feynman’s ideas in quantum physics in the realm of stochastic evolution equations.

**References**


